There Are Infinitely Many Prime Twins

R. F. Arenstorf

Department of Mathematics, Vanderbilt University, Nashville, TN 37240; USA

Abstract. A proof of the twin-prime conjecture, even in the stronger form of Hardy and Littlewood, namely that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{\substack{p<N \\ p,p+2 \text{ both prime}}} \log p \cdot \log(p+2) \;=\; B_2 \;>\; 0,$$

is presented using methods from classical analytic number theory.

1 Introduction

A natural generating function for the twin-prime problem is the Dirichlet series

$$T(s) := \sum_{n>3} \Lambda(n-1)\Lambda(n+1)n^{-s} \quad (\Re e\ s > 1).$$

Here we will show that $T(s) - B_2(s-1)^{-1}$ can be continuously extended onto the line $\Re e\, s = 1$, (Th.1). This enables application of the powerful complex Tauberian theorem of Wiener and Ikehara (1931) and then almost immediately yields our main result, (Th.2).

The ideas basic to our proof of Th.1 are plain: We are striving to exhibit a relation between $T(s)$ and the Riemann Zeta function $\zeta(s)$; motivated by a proof of the Prime Number Theorem along the above lines [where $T(s)$ is replaced by $[\log \zeta(s)]'$ and only $\zeta(s) \neq 0$ on $\Re e\, s = 1$ needs to be known]. Now, $T(s)$ can be obtained by a "kind of sieving" from the Dirichlet series of $[\log \zeta(s)]''$, which leads to the arithmetic formula (2) in Lemma 1 for the "characteristic" function $\Lambda(n-1)\Lambda(n+1)$ of the prime powers with difference 2.

This important formula leads to a representation of $T(s)$ by a double series over all odd squarefree m in $\mathbb{Z}^+$ and over all n in $\mathbb{Z}^+$ satisfying the quadratic congruence $4n^2 \equiv 1 \bmod m$. The number of such n in any interval of length m is a known multiplicative function of m, which heuristically leads to the above singular term for $T(s)$ with, remarkably, the same constant B_2 as made plausible by the classical circle method of Hardy & Littlewood. But splitting off such a "main term" and studying the remainder turned out be counterproductive. In this context a look at some papers of P. Turan [5], [6] is informative.

The individual solutions of the above quadratic congruence can be characterized variously using the prime factorization of m, or exponential sums (similar to Kloostermann's, say) or Dirichlet characters modulo m; a.o. However, none of these customary (and mainly implicit) descriptions turned out to be useful for our purpose. Instead we developed an explicit parameter representation for these solutions (Lemma 2); by a reduction to pairs of linear congruences. Introducing this representation essentially converts $T(s)$ into a quadruple series over four (controllable) parameters a, c, k, ℓ, say; each ranging over the positive odd integers and jointly subject to the single constraint $C_2 : a\ell - ck = 2$, however.

We enforce C_2 as usual by multiplying the terms of $T(s)$ with an appropriate trigonometric integral, but we do not then switch this integration with the quadruple summation. Instead we first modify the integral for C_2 (in essence) by replacing the cosine function under the integral by the inverse transform of its Mellin-transform. This yields an absolutely convergent double integral (Lemma 5), and we then switch the two complex integrations with the free quadruple summation.

It is indicative of the binary nature of the twin-prime problem that an exact treatment of the resulting (double) integral here is feasible; in contrast to, for instance, the ternary Goldbach-Vinogradov problem, where the circle method affords (only) an approximate, yet effective treatment of the relevant integral. Namely, by design, the quadruple series appearing under the (double) integral can be "summed in closed form"; i.e. it can be expressed (in simplified description) as

$$\zeta(w)M''(w,z)\zeta(z), \text{ where } M''(w,z) := \sum_{k,\ell} \mu(2k\ell)\,[\log k\ell]^2\, k^{-w}\ell^{-z}, \quad (u, x > 1)$$

is a (double) Dirichlet series in two complex variables, which is characteristic for the twin-prime problem (cf. Lemma 1). Its needed analytic continuation will be obtained from

$$M''(w,z) = \left(\frac{\partial}{\partial w} + \frac{\partial}{\partial z}\right)^2 \left[\zeta_0^{-1}(w)P(w,z)\zeta_0^{-1}(z)\right] \quad \text{with} \quad \zeta_0(z) := \sum_{n \text{ odd}} n^{-z}$$

and

$$P(w,z) := \prod_{p>2} \left[1 - (p^w - 1)^{-1}(p^z - 1)^{-1}\right] \quad \text{for} \quad u > 0,\, x > 0 \;\; \& \;\; u + x > 1.$$

This Euler product in two variables converges absolutely and compact uniformly, so that $M''(w,z)$ is holomorphic in each variable for $u > 3/4$ & $x > 3/4$ say, except at the zeros of $\zeta(w)\zeta(z)$. The function M'' of two complex variables replaces here the Dirichlet L-functions with residue characters so prevalent in analytic prime number theory. For our application it suffices to make use of a known zero-free region for $\zeta(w)$ of the form

$$u \geq 1 - h(v) \quad \text{with} \quad h(v) = c[\log(2 + |v|)]^{-a}, \quad (v \text{ in } \mathbb{R}) \quad \text{and} \quad a < 1;$$

we choose $a = 4/5$ (to avoid factors like $\log\log(3+|v|)$), though better choices are available.

The development outlined so far yields, in essence, a representation of $T(s)$ by iterated integrals of Mellin-Barnes type involving besides $M''(w,z)$ several gamma and zeta functions a.o., which is valid primarily for $\Re e\, s > 2$. In the remaining (major) part of this work we construct its analytic continuation to $\Re e\, s \geq 1$. The derivations are mainly technical, employ convergence enhancing tools and are concerned with demonstrating existence of (up to 3-fold) iterated contour integrals. These parameter-dependent integrals (with movable singularities in their integrands) are taken along vertical straight lines, and ultimately along "zero-free curves" such as $N_w : u = 1 - h(v)$, $(-\infty < v < \infty)$. They are transformed by decomposition, shifting of contours, picking up residues or exchange of integrations, etc. For each manipulation an allowable domain for the variables, to maintain convergence, is determined; while these manipulations overall are designed to extend the domain of validity of the integral representations stepwise to their goal. Finally, for the case of an only "conditionally convergent" twice iterated integral occurring in our Main Lemma, a (lengthy) series of very detailed careful estimates is given to establish its actual convergence.

This work is the outcome of about twenty years of "on and off" search and research on this and the related binary Goldbach problem; in the interim having been lured onto various misleading paths or frustrated by (for me) insurmountable difficulties, before ultimately recognizing and constructing a workable approach.

2 The Characteristic Euler Product

Let

$$T(s) := \sum_{n>3} \Lambda(n-1)\Lambda(n+1)n^{-s} \quad (\Re e\ s > 1)$$

and

$$B_2 = 2\prod_{p>2}\left[1-(p-1)^{-2}\right] \approx 1.320, \tag{1}$$

where $\Lambda(\cdot)$ is the von Mangoldt function and B_2 the Twin Prime Constant.

Theorem 1 *The function $T_1(s) := T(s) - B_2/(s-1)$ on $\Re e\ s > 1$ has a continuous extension to $\Re e\ s \geq 1$; i.e. $\lim_{\sigma\searrow 1} T_1(\sigma + \mathrm{i}t) =: f_1(t)$ uniformly for $t \in [-T,T]$ and every $T > 0$.*

A proof of this theorem will be supplied in what follows; it requires several auxiliary results which we derive at first.

Lemma 1 *For* **even** $n > 3$:

$$\Lambda(n-1)\Lambda(n+1) = \frac{1}{2} \sum_{m|n^2-1} \mu(m) \log^2 m. \qquad (2)$$

Namely, with $\mu(\cdot)$ denoting the Möbius function,

$$\sum_{mn=k} \Lambda(m)\Lambda(n) = \Lambda(k)\log k + \sum_{m|k} \mu(m) \log^2 m \quad (k \geq 1),$$

as follows by a simple calculation from

$$\Lambda(n) = -\sum_{d|n} \mu(d) \log d \quad \& \quad \sum_{d|m} \Lambda(d) = \log m.$$

Now taking $k = ac$ with $a > 1, c > 1$ & $(a, c) = 1$, we have

$$\sum_{mn=k} \Lambda(m)\Lambda(n) = 2\Lambda(a)\Lambda(c) \quad \& \quad \Lambda(k) = 0$$

and choosing $a = n - 1, c = n + 1$ implies (2).

We also need a special parameter representation for the solutions of a quadratic congruence:

Lemma 2 *Let* $m \in \mathbb{Z}^+$ *be* **odd** *and squarefree. The* **even** *numbers* $n \in \mathbb{Z}^+$ *satisfying* $n^2 \equiv 1 \bmod m$ *are represented exactly once by the expression*

$$n = \tfrac{1}{2}(a\ell + ck),$$

when a, c, k, ℓ *vary over all* **odd** *integers in* $\mathbb{Z}^+$ *satisfying* $k\ell = m$ *and* $a\ell - ck = 2$.

Proof: 1.) Let m have r prime factors. There are exactly 2^r distinct solutions $x = x_j$ in $\mathbb{Z}$, $(j = 1, \ldots, 2^r)$ of $4x^2 \equiv 1 \bmod m$ & $1 \leq x \leq m$, since m is squarefree; and so the numbers n of the lemma are given exactly by $n = 2x_j + 2hm$, $(j = 1, \ldots, 2^r; h = 0, 1, 2, \ldots)$.

2.) Let k be a divisor of m and $\ell := m/k$. Then $(k, \ell) = 1$, and since also $(2, k) = (2, \ell) = 1$, there exist a unique $b \in \mathbb{Z}$ with

$$2b \equiv 1 \bmod k, \quad 2b \equiv -1 \bmod \ell \;\; \& \;\; 1 \leq b \leq m.$$

Then

$$2b + 1 = a_0\ell, \;\; 2b - 1 = c_0k \;\; \& \;\; 4b^2 = 1 + a_0c_0m \equiv 1 \bmod m$$

with **odd** a_0, c_0 in $\mathbb{Z}^+$ and $a_0\ell - c_0k = 2$. Hence by 1.) $b = x_j$ for a unique $j = j(k)$, $(1 \leq j(k) \leq 2^r)$. Also

$$(2x_j - 1, m) = (c_0k, \ell k) = k \quad \text{for } j = j(k), \text{ since } (c_0, \ell) = 1.$$

Now m has exactly $d(m) = 2^r$ (distinct) divisors k, and so the associated 2^r numbers x_j for $j = j(k)$ are all distinct. This implies by 1.) that the index $j(k)$ assumes every (integral) value j in $1 \le j \le 2^r$ exactly once, when k runs through the divisors of m.

3.) Select k & ℓ with $k\ell = m$. Then by 2.) for $j = j(k)$, we have

$$2x_j = \tfrac{1}{2}(a_0\ell + c_0k) \;\;\&\;\; a_0\ell - c_0k = 2 \quad \text{with } \textbf{odd} \;\; a_0, c_0 \text{ in } \mathbb{Z}^+.$$

Now all **odd** a, c in $\mathbb{Z}^+$ satisfying $a\ell - ck = 2$ are given by

$$a = a_0 + 2hk \;\;\&\;\; c = c_0 + 2h\ell \;\;\text{with}\;\; h \ge 0 \text{ in } \mathbb{Z},$$

since $(k,\ell) = 1$ and $c_0k = 2b - 1 \le 2m - 1 < 2k\ell$ implies $c_0 < 2\ell$. Then

$$n := \tfrac{1}{2}(a\ell + ck) = \tfrac{1}{2}(a_0\ell + c_0k) + 2hm = 2x_j + 2hm \in \mathbb{Z}^+, \quad (h \ge 0)$$

is **even**, satisfies $n^2 \equiv 1 \bmod m$, and is represented as stated in the lemma (with $a\ell - ck = 2; a, c$ **odd** in $\mathbb{Z}^+$). When now k runs through the divisors of m we obtain by 2.) every n listed under 1.) exactly once. This proves Lemma 2. □

Corollary *Let $h(n) \in \mathbb{C}$ be defined for $n \in \mathbb{Z}^+$ and $\mu(2m) \neq 0$, then*

$$\sum_{\substack{n>1,\,\text{even} \\ n^2\equiv 1 \bmod m}} h(n) = \sum_{\substack{k|m \\ (\ell=m/k)}} \sum_{\substack{a,c \\ a\ell-ck=2}}^{\circ} h\left(\frac{a\ell+ck}{2}\right), \quad \textit{where} \;\; \sum_a^{\circ} \;\; \textit{means} \;\; \sum_{a>0,\,\text{odd}}, \tag{3}$$

if the first series is abs. conv. (= absolutely convergent).

Next we introduce the function $M(w,z)$ by the abs. conv. series

$$M(w,z) := \sum_m^{\circ} \mu(m) \sum_{k\ell=m} k^{-w}\ell^{-z} = \sum_{k,\ell}^{\circ} \mu(k\ell)k^{-w}\ell^{-z} \quad \text{for } u>1,\ x>1 \tag{4}$$

$(w = u + \mathrm{i}v,\ z = x + \mathrm{i}y)$ and extend it analytically by the following calculation

$$\begin{aligned} M(w,z) &= \sum_{\substack{k,\ell \\ (k,\ell)=1}}^{\circ} \mu(k)\mu(\ell)k^{-w}\ell^{-z} \\ &= \sum_{k,\ell}^{\circ} \mu(k)\mu(\ell)k^{-w}\ell^{-z} \sum_{d|k,d|\ell} \mu(d) \\ &= \sum_d^{\circ} \mu(d) \sum_{k'}^{\circ} \mu(k'd)(k'd)^{-w} \sum_{\ell'}^{\circ} \mu(\ell'd)(\ell'd)^{-z} \\ &= \sum_d^{\circ} \mu^3(d)d^{-w-z} \sum_{\substack{k \\ (k,d)=1}}^{\circ} \mu(k)k^{-w} \sum_{\substack{\ell \\ (\ell,d)=1}}^{\circ} \mu(\ell)\ell^{-z} \\ &= \sum_d^{\circ} \mu(d)d^{-w-z} \prod_{p \nmid 2d} \left[(1-p^{-w})(1-p^{-z})\right] \end{aligned}$$

$$
\begin{aligned}
&= \zeta^{-1}(w)\zeta^{-1}(z)(1-2^{-w})^{-1}(1-2^{-z})^{-1} \\
&\quad \cdot \sum_d{}^{\circ} \mu(d) d^{-w-z} \prod_{p|d} \left[(1-p^{-w})(1-p^{-z})\right]^{-1} \\
&= \zeta_0^{-1}(w)\zeta_0^{-1}(z) P(w,z), \qquad (5)
\end{aligned}
$$

where

$$
\zeta_0(z) := (1-2^{-z})\zeta(z) = \sum_k{}^{\circ} k^{-z}
$$

and the Euler product

$$
P(w,z) := \prod_{p>2} \left[1 - \frac{p^{-w-z}}{(1-p^{-w})(1-p^{-z})}\right] \qquad (6)
$$

is abs. conv. on $u, x > 0$ & $u + x > 1$ and so a holomorphic function of w & z there, and $\zeta(\cdot)$ is the Riemann Zeta function. Hence $M(w,z)$ is holomorphic there except at the zeros of $\zeta(w)\zeta(z)$. We also need

$$
M''(w,z) := \sum_m{}^{\circ} \mu''(m) \sum_{k\ell=m} k^{-w}\ell^{-z} \qquad (7)
$$

with $\mu''(m) := \mu(m)\log^2 m, \quad (u, x > 1)$. Then

$$
\begin{aligned}
M''(w,z) &= \left(\frac{\partial}{\partial w} + \frac{\partial}{\partial z}\right)^2 M(w,z) \quad =: \quad \hat{D}^2 M \\
&= \hat{D}^2(GP) \quad = \quad G''P + 2G'P' + GP'', \qquad (7a)
\end{aligned}
$$

where

$$
G := G(w,z) := \zeta_0^{-1}(w)\zeta_0^{-1}(z), \;\; P := P(w,z), \;\; P' := \hat{D}P, \;\; P'' := \hat{D}^2 P \text{ etc.} \qquad (7b)
$$

and so $M''(w,z)$ is a holomorphic function of w & z on $u > 0$, $x > 0$ & $u+x > 1$ except at the zeros of $\zeta(w)\zeta(z)$.

Lemma 3 *$M(w,z)$ and $M''(w,z)$ are holomorphic functions of w & z on $u \geq 1 - 2h(v)$ & $x \geq 1 - 2h(y)$ with $h(t) := c[\log(2+|t|)]^{-4/5} > 0$ for $t \in \mathbb{R}$ and $c < \frac{1}{8}\log 2$ a constant. Also*

$$
M''(w,z) \ll [\log(2+|v|)\cdot\log(2+|y|)]^3 \quad \text{there and} \quad M''(1,1) = 4B_2.
$$

Proof: It is known that $\zeta(w) \neq 0$ and $\zeta^{-1}(w)$ & $\zeta'(w)\zeta^{-1}(w) \ll h^{-1}(v)$ as $v \to \pm\infty$ and $u \geq 1 - 3h(v)$, say [as implied by Titchmarsh [4], Th. 3.11 and p.114]. Setting $f(w) := \zeta_0^{-1}(w)$ this yields $f'(w) \ll h^{-2}(v)$ and $f''(w) \ll h^{-3}(v)$ for $u \geq 1 - 2h(v)$ and $v \in \mathbb{R}$. Since $G(w,z) = f(w)f(z)$ we get

$$
G' = f'(w)f(z) + f(w)f'(z), \quad G'' = f''(w)f(z) + 2f'(w)f'(z) + f(w)f''(z) \qquad (7c)
$$

and so $G, G', G'' \ll h^{-3}(v)h^{-3}(y) \ll [\log(2+v)\cdot\log(2+y)]^3$, $(v, y \geq 0)$ amply. Also $h(t) < \frac{1}{8}$ for $t \geq 0$ gives $u \geq 3/4$ & $x \geq 3/4$ in the above domain for w & z, and so P, P', and P'' are bounded there, by (6). Now (7a) yields the stated estimate for M''. Finally, since $\zeta_0(w)$ has a simple pole at $w = 1$ with residue $1/2$, we get

$$f(1) = 0, \quad f'(1) = 2, \quad \text{and so} \quad G(1,1) = 0 = G'(1,1) \text{ \& } G''(1,1) = 8$$

from our preceeding formulas, and then $M''(1,1) = 8P(1,1) = 4B_2$ by eqs. (6) and (1); completing the proof of Lemma 3. □

Lemma 4 *Let* $C(w) := \frac{1}{\pi}\int_0^\pi (1+\cos t)t^{-w}\,dt$ *for* $u < 1$*. Then also*

$$C(w) = \frac{2\pi^{-w}}{1-w} + \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!}\frac{\pi^{2n-w}}{2n+1-w} =: \pi^{-w}\tilde{C}(w) \quad (say)$$

and $C(w) + \frac{2/\pi}{w-1}$ *is holomorphic on* $u < 3$*; and* $C(w) \ll |v|^{-3}$ *for* $|u| < 3$ & $|v| > 1$*.*

Proof: Inserting the power series of $\cos t$ and integrating termwise gives the first part of the statement. And three partial integrations yield

$$\begin{aligned} \pi C(w) &= \frac{1}{1-w}\int_0^\pi t^{1-w}\sin t\,dt = \frac{-1}{(1-w)(2-w)}\int_0^\pi t^{2-w}\cos t\,dt \quad \text{for } u < 3 \\ &= \frac{1}{(1-w)(2-w)(3-w)}\left(\pi^{3-w} - \int_0^\pi t^{3-w}\sin t\,dt\right) \quad \text{for } u < 5, \end{aligned}$$

showing poles at $w = 1$ & $w = 3$ (but not at $w = 2$, where $C(w)$ is holomorphic). Then for $|u| < 3$ & $|v| > 1$

$$|C(w)| < 3\pi^5(1+\pi)\cdot\prod_{k=1}^{3}(|k-u|+|v|)^{-1} < 13\pi^5\cdot|v|^{-3}; \text{ q.e.d.}$$

Alternatively,

$$\begin{aligned} \tilde{C}(w) &= \int_0^1 (1-\cos\pi x)(1-x)^{-w}\,dx \\ &= -\sum_{n=1}^{\infty}\frac{(-1)^n\pi^{2n}}{(2n)!}B(2n+1, 1-w), \quad (u < 1) \\ &= -\sum_{n=1}^{\infty}\frac{(i\pi)^{2n}\Gamma(1-w)}{\Gamma(1-w+2n+1)} \\ &= -\sum_{n=1}^{\infty}(i\pi)^{2n}\cdot\prod_{k=0}^{2n}(k+1-w)^{-1} \quad \ll \quad |v|^{-3}. \end{aligned}$$

□

We mention that a more natural definition of $C(w)$ without the 1 under the integral would yield $C(w) \ll |v|^{-1}$ only, which is too weak for our applications.

Lemma 5 *Let*

$$E(x,y) := \frac{1}{\pi}\int_0^\pi (1+\cos t)\cos\left[(x-y)t\right] dt \quad \textit{for } x>0,\ y>0. \tag{8}$$

Then also

$$\begin{aligned} E(x,y) &= \int_{(-3/4)}^{*} \Gamma(w)x^{-w}dw \int_{(1/2)}^{*} \Gamma(\tilde{w})y^{-\tilde{w}}C(w+\tilde{w})\cos\left[\tfrac{\pi}{2}(w-\tilde{w})\right] d\tilde{w} \\ &\quad + \int_{(1/2)}^{*} \Gamma(w)y^{-w}C(w)\cos\tfrac{\pi w}{2}\, dw, \end{aligned}$$

where the individual integrals and the iterated integral are abs. conv.. Here and in the following our notation means

$$\int_{(a)}^{*} f(w)dw := \frac{1}{2\pi \mathrm{i}}\int_{a-\mathrm{i}\infty}^{a+\mathrm{i}\infty} f(w)dw \ \ \&\ \textbf{abs. conv.}; \textit{ i.e.} \int_{-\infty}^{\infty} |f(a+\mathrm{i}v)|dv < \infty. \tag{9}$$

Proof: We have for $x\in\mathbb{R}$

$$e^{-\mathrm{i}x} = 1 + \int_{(-3/4)}^{*} \Gamma(w)(\mathrm{i}x)^{-w}dw \quad \text{with } (\pm\mathrm{i})^{-w} = e^{\mp \mathrm{i}\pi w/2} \ll e^{\pi|v|/2},$$

$$\begin{aligned} e^{-\mathrm{i}x}e^{\mathrm{i}y} &= 1 + \int_{(-3/4)}^{*} \Gamma(w)\left[(\mathrm{i}x)^{-w} + (-\mathrm{i}y)^{-w}\right] dw \\ &\quad + \int_{(-3/4)}^{*}\int_{(-3/4)}^{*} \Gamma(w)\Gamma(\tilde{w})(\mathrm{i}x)^{-w}(-\mathrm{i}y)^{-\tilde{w}}dwd\tilde{w} \end{aligned}$$

for $x,y\in\mathbb{R}$; and also the double integral is abs. conv. So for $x,y>0$

$$\begin{aligned} \cos(x-y) &= 1 + \int_{(-3/4)}^{*} \Gamma(w)(x^{-w}+y^{-w})\cos\tfrac{\pi w}{2}dw \\ &\quad + \int_{(-3/4)}^{*}\int_{(-3/4)}^{*} \Gamma(w)\Gamma(\tilde{w})x^{-w}y^{-\tilde{w}}\cos\left[\tfrac{\pi}{2}(w-\tilde{w})\right] dwd\tilde{w}. \end{aligned}$$

Now with Lemma 4, by abs. conv. of all occurring integrals

$$\begin{aligned} E(x,y) &= C(0) + \int_{(-3/4)}^{*} \Gamma(w)(x^{-w}+y^{-w})C(w)\cos\tfrac{\pi w}{2}dw \\ &\quad + \int_{(-3/4)}^{*} \Gamma(w)x^{-w}dw \int_{(-3/4)}^{*} \Gamma(\tilde{w})y^{-\tilde{w}}C(w+\tilde{w})\cos\left[\tfrac{\pi}{2}(w-\tilde{w})\right] d\tilde{w} \end{aligned}$$

and this yields the stated representation of $E(x,y)$ by shifting the line of integration of the last (inner) integral to the right over the pole $\tilde{w}=0$ and then doing the same with the remaining single integral. This is justified using

$$\begin{aligned} &\Gamma(w) \ll |v|^{u-1/2}e^{-\pi|v|/2},\ C(w) \ll |v|^{-3} \quad \text{for } -2<u<3 \ \ \&\ \ |v|>1 \\ &\text{and } \cos\tfrac{\pi w}{2} \ll e^{\pi|v|/2}; \end{aligned} \tag{10}$$

and so

$$\int_{(-3/4)} \int_{(\tilde{u})} \left|\Gamma(w)\Gamma(\tilde{w})C(w+\tilde{w})\cos\left[\tfrac{\pi}{2}(w-\tilde{w})\right]\right| dv d\tilde{v}$$
$$\ll \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (1+|v|)^{-5/4}(1+|\tilde{v}|)^{\tilde{u}-1/2}(1+|v+\tilde{v}|)^{-3} dv d\tilde{v}$$
$$\ll \int_{-\infty}^{\infty} (1+|v|)^{-5/4}\, dv \cdot \int_{-\infty}^{\infty} (1+|t|)^{-3}\, dt < \infty \text{ for } -3/4 \le \tilde{u} \le 1/2, \quad (10')$$

which implies the abs. convergence of all double integrals and then of all iterated and single integrals involved in this calculation; proving Lemma 5. □

3 Transformation of the Generating Function

We now introduce the generating function

$$T(s,\delta) := \sum_{n>3} \Lambda(n-1)\Lambda(n+1)(n^2-1)^{-s} e^{-n\delta} \cosh\delta, \quad (s=\sigma+\mathrm{i}t;\ \delta>0) \quad (11)$$

so that by Abel's Theorem for power series

$$\lim_{\delta\searrow 0} T(s,\delta) = T(s,0) := \sum_{n>3} \Lambda(n-1)\Lambda(n+1)(n^2-1)^{-s} \quad \text{for } \sigma > 1/2. \quad (12)$$

Later we use $T(s/2,0)$, since this differs from $T(s)$ in (1) by a function holomorphic on $\sigma > 0$ (at least). By Lemma 1 and always for $\delta > 0$,

$$\begin{aligned} 2T(s,\delta) &= \sum_{n>3,\,\mathrm{even}} (n^2-1)^{-s} e^{-n\delta}\cosh\delta \cdot \sum_{m|n^2-1} \mu''(m) \quad \left(\mu''(n) := \mu(n)\log^2 n\right) \\ &= -3^{-s}e^{-2\delta}\mu''(3)\cosh\delta + \\ &\qquad \sum_{m>0,\,\mathrm{odd}} \mu''(m) \sum_{\substack{n>1,\,\mathrm{even} \\ n^2\equiv 1 \bmod m}} (n^2-1)^{-s} e^{-n\delta}\cosh\delta \\ &=: A(s,\delta) + \sum_m{}^{\circ} \mu''(m) R(s,\delta,m), \quad \text{(say) for } \sigma > 1/2, \end{aligned} \quad (13)$$

where by eq. (3) (with $n+1 = a\ell$ & $n-1 = ck$) and by abs. conv.

$$\begin{aligned} R(s,\delta,m) &= \frac{1}{2}\sum_{k\ell=m} \sum_{\substack{a,c \\ a\ell-ck=2}}{}^{\circ} (a\ell ck)^{-s}\left[e^{-a\ell\delta}+e^{-ck\delta}\right] \qquad (\sigma>1; m \text{ odd}) \\ &= \frac{1}{2}\sum_{k\ell=m} m^{-s} \sum_{a,c}{}^{\circ} (ac)^{-s}\left[e^{-a\ell\delta}+e^{-ck\delta}\right] \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{\mathrm{i}\phi\left(1-\frac{a\ell-ck}{2}\right)}\, d\phi \\ &= m^{-s}\sum_{k\ell=m}\sum_{a,c}{}^{\circ} (ac)^{-s} e^{-a\ell\delta} \frac{1}{\pi}\int_0^{\pi} \cos\phi \cos\left[\tfrac{1}{2}(a\ell-ck)\phi\right]\, d\phi. \end{aligned} \quad (14)$$

Again for **odd** squarefree $m > 0$ and $\sigma > 1$ let now

$$\begin{aligned} R_0(s,\delta,m) \quad &:= \quad m^{-s}\sum_{k\ell=m}\sum_{a,c}{}^{\circ}(ac)^{-s}e^{-a\ell\delta}\frac{1}{\pi}\int_0^{\pi}\cos\left[\tfrac{1}{2}(a\ell-ck)\phi\right]\,d\phi \qquad (15)\\ &= \quad m^{-s}\sum_{k\ell=m}\sum_{\substack{a,c\\ a\ell-ck=0}}{}^{\circ}(ac)^{-s}e^{-a\ell\delta} = m^{-s}\sum_{k\ell=m}\sum_{h}{}^{\circ}(h^2m)^{-s}e^{-hm\delta}\\ &= \quad m^{-2s}d(m)\sum_{h}{}^{\circ}h^{-2s}e^{-hm\delta}, \text{ since here } (k,\ell)=1 \text{ and} \end{aligned}$$

then $a\ell = ck$ implies $a = hk$ & $c = h\ell$ with **odd** $h > 0$ for **odd** ac, and reverse. Then in analogy to $T(s,\delta)$ in (13) we consider for $\sigma > 1/2$

$$\begin{aligned} 2T_0(s,\delta) \quad &:= \quad \sum_{m}{}^{\circ}\mu''(m)R_0(s,\delta,m) = \sum_{m}{}^{\circ}\mu''(m)d(m)m^{-2s}\sum_{h}{}^{\circ}h^{-2s}e^{-hm\delta}\\ &= \quad \sum_{k}{}^{\circ}k^{-2s}e^{-k\delta}\sum_{m|k}\mu''(m)d(m) \quad ; \qquad (16) \end{aligned}$$

i.e.,

$$\begin{aligned} \lim_{\delta\searrow 0} T_0(s/2,\delta) \quad &= \quad T_0(s/2,0) = \frac{1}{2}\sum_{k}{}^{\circ}k^{-s}\sum_{m|k}\mu''(m)d(m) \quad (\sigma > 1)\\ &= \quad \frac{1}{2}\sum_{m}{}^{\circ}\mu''(m)d(m)\sum_{h}{}^{\circ}(mh)^{-s} = \frac{1}{2}M''(s,s)\zeta_0(s)\\ &=: \quad \frac{B_2}{s-1} + B(s)\ , \quad \text{(say)} \qquad (17) \end{aligned}$$

by (7) for $w = z = s$, and $B(s)$ is holomorphic on $\sigma \geq 1$ by Lemma 3.

Combining eqs. (14), (15), and (8) we now obtain

$$R_2(s,\delta,m) := R(s,\delta,m) + R_0(s,\delta,m) = m^{-s}\sum_{k\ell=m}\sum_{a,c}{}^{\circ}(ac)^{-s}e^{-a\ell\delta}E\left(\tfrac{a\ell}{2},\tfrac{ck}{2}\right) \qquad (18)$$

and from eqs. (13), (16) and (18)

$$2T_2(s,\delta) := 2T(s,\delta) + 2T_0(s,\delta) - A(s,\delta) = \sum_{m}{}^{\circ}\mu''(m)R_2(s,\delta,m), \qquad (19)$$

and all these series are abs. conv. for $\sigma > 1$. Using [with notation from (9)]

$$e^{-x} = \int_{(b)}^{*}\Gamma(\tau)x^{-\tau}\,d\tau \quad \text{for } x > 0 \;\&\; b > 0 \text{ and letting } x = a\ell\delta$$

we obtain from (18) by allowed exchange of integration and summations

$$R_2(s,\delta,m) = m^{-s}\int_{(b)}^{*}\delta^{-\tau}\Gamma(\tau)\sum_{k\ell=m}\sum_{a,c}{}^{\circ}(ac)^{-s}(a\ell)^{-\tau}E\left(\tfrac{a\ell}{2},\tfrac{c\ell}{2}\right)\,d\tau, \quad (\sigma > 1)$$

and then from (19), again by abs. conv. for $\sigma > 1$, and taking $b = 3/2$

$$\begin{aligned} 2T_2(s,\delta) &= \int_{(b)}^{*} \delta^{-\tau}\Gamma(\tau)\sum_m{}^{\circ}\mu''(m)m^{-s}\cdot\sum_{k\ell=m}\sum_{a,c}{}^{\circ}(ac)^{-s}(a\ell)^{-\tau}E\left(\tfrac{a\ell}{2},\tfrac{ck}{2}\right)\,d\tau \\ &=: \int_{(3/2)}^{*}\delta^{-\tau}\Gamma(\tau)D(s,\tau)d\tau \quad \text{(say)};\ (\sigma>1,\delta>0). \end{aligned} \tag{20}$$

4 Integral Representation of $T(s,\delta)$

Substituting for $E(\cdot,\cdot)$ the two expressions from Lemma 5 in (20) yields

$$\begin{aligned} D(s,\tau) &:= \sum_{k,\ell}{}^{\circ}\mu''(k\ell)k^{-s}\ell^{-s-\tau}\sum_{a,c}{}^{\circ}a^{-s-\tau}c^{-s}E\left(\tfrac{a\ell}{2},\tfrac{ck}{2}\right) \\ &=: D_1(s,\tau)+D_2(s,\tau) \quad \text{(say)},\ (\sigma>1,\ \Re e\,\tau=3/2) \end{aligned} \tag{21}$$

where then by definition in (7) and with $L_0(w,z) := \zeta_0(w)\zeta_0(z)M''(z,w)$ for $u,x>1$,

$$D_1(s,\tau) = \int_{(1/2)}^{*}\Gamma(w)2^wC(w)\cos\tfrac{\pi w}{2}L_0(s+\tau,s+w)\,dw,\ (\sigma>1/2,\ \Re e\,\tau=3/2) \tag{21a}$$

and

$$D_2(s,\tau) = \int_{(-3/4)}^{*}\Gamma(w)2^w dw\int_{(1/2)}^{*}\Gamma(\tilde w)2^{\tilde w}C(w+\tilde w)\cos\left[\tfrac{\pi}{2}(w-\tilde w)\right]L_0(s+\tau+w,s+\tilde w)d\tilde w, \tag{21b}$$

the exchanges of integration and summations being justified by abs. conv.. Even the last iterated integral is abs. conv., as follows from the estimation (10′), since L_0 here is bounded on the lines of integration when $\sigma > 1/2$ & $\Re e\,\tau = 3/2$.

Introducing

$$L(w,z) := \zeta_2(w)\zeta_2(z)M''(w,z) \quad\text{with}\quad \zeta_2(w) := 2^w\zeta_0(w) = (2^w-1)\zeta(w) \tag{22}$$

we have

$$L(w,z) = L(z,w) = 2^{w+z}L_0(w,z);$$

and the analytic properties of $L(w,z)$ follow from (5), (6), (7a,b) and Lemma 3. Writing

$$R_w := \left\{w = u+\mathrm{i}v \text{ with } u \ge 1-h(v),\ v\in\mathbb{R} \text{ and } h(v) = c\cdot[\log(2+|v|)]^{-4/5}\right\} \tag{23}$$

and $N_w := \partial R_w$ for the boundary curve $u = 1-h(v)$, $(-\infty < v < \infty)$ of this domain in the w-plane (and correspondingly R_z, N_z in the z-plane), then

$L(w,z)$ is holomorphic in each variable on $\{R_w,\, w\neq 1\}\times\{R_z,\, z\neq 1\}$ by (22) and Lemma 3 and has simple poles at $w=1$ and at $z=1$ with

$$\begin{aligned} \operatorname*{Res}_{(w=1)} L(w,z) &= \zeta_2(z)M''(1,z) =: R(z), \quad \text{say} \quad (z\neq 1) \\ \operatorname*{Res}_{(z=1)} L(w,z) &= R(w) \ \ \& \ \operatorname*{Res}_{(w=1)} R(w) = M''(1,1) = 4B_2. \end{aligned} \tag{24}$$

Also, $L(w,z)$ is bounded on the vertical lines $\Re e\, w = a$ & $\Re e\, z = b$ when $a,b>1$, and so are $R(w)$ & $R(z)$. Observing that

$$\zeta(z) \ll y^{(1-x)/2+\epsilon} \quad \text{for } 1/2\le x\le 1 \ \ \& \ y>y_0>3 \tag{25}$$

and arbitrarily small $\epsilon>0$, we obtain with Lemma 3

$$R(z) \ll y^{h(y)+\epsilon}\log^3(2+y) \ll y^{2\epsilon} \quad \text{for } 1-h(y)\le x\le 1 \ \& \ y>y_0\,, \tag{26}$$

and by (10) and (22), for $\Re e\, \tilde{w}>1$ and fixed s in $1/4<\sigma<3/4$:

$$\Gamma(z-s)L(\tilde{w},z)C(z-s)\cos\left[\tfrac{\pi}{2}(z-s)\right] \ll y^{x-\sigma-\frac{1}{2}+2\epsilon-3} \ll y^{-2} \tag{27}$$

for $1-h(y)\le x<5/4<1+\sigma$ & $y\gg 1$. This is needed for (21a,b).

To achieve the analytic continuation of $D(s,\tau)$ from (21) & (21a,b) and thereby of $T_2(s,\delta)$ in (20), from $\sigma>1$ to $\sigma\ge 1/2$, we shall perform some careful calculations of multiple Mellin-Barnes type integrals. First we elaborate on some implications for iterated integrals of our $*$ notation introduced in eq. (9).

Lemma 6 *For suitable functions f, g and h let*

$$F(z) := \int_{\mathbb{R}} f(z,w)\,dv = \int_{\mathbb{R}} g(z,w)\,dv + h(z) =: G(z)+h(z) \tag{a}$$

and

$$\int_{\mathbb{R}}^{*} F(z)\,dy \ \ \& \ \ \int_{\mathbb{R}}^{*} h(z)\,dy \quad \textit{both exist.} \tag{b}$$

Then

$$\int_{\mathbb{R}}^{*} dy\int_{\mathbb{R}} f(z,w)\,dv = \int_{\mathbb{R}}^{*} dy\int_{\mathbb{R}} g(z,w)\,dv + \int_{\mathbb{R}}^{*} h(z)\,dy. \tag{c}$$

Proof: (b) means (only) that

$$\int_{\mathbb{R}} |F(z)|\,dy = \int_{\mathbb{R}} dy\left|\int_{\mathbb{R}} f(z,w)\,dv\right| < \infty \ \ \& \ \ \int_{\mathbb{R}} |h(z)|\,dy<\infty\,,$$

and this implies with (a) amply that

$$\int_{\mathbb{R}} G(z)\,dy = \int_{\mathbb{R}} (F(z)-h(z))\,dy = \int_{\mathbb{R}} F(z)\,dy - \int_{\mathbb{R}} h(z)\,dy. \tag{d}$$

But (c) states more, namely that

$$\int_{\mathbb{R}}^{*} G(z)\,dy \quad \text{exists; i.e.} \quad \int_{\mathbb{R}} |G(z)|\,dy < \infty,$$

and this also follows from (b):

$$\int_{\mathbb{R}} |G(z)|\,dy = \int_{\mathbb{R}} |F(z) - h(z)|\,dy \leq \int_{\mathbb{R}} (|F(z)| + |h(z)|)\;dy < \infty,$$

Now (d) implies (c). We note that the iterated integrals in (c) need not be abs. conv. This procedure leading from (a) to (c) will be applied repeatedly, where $h(z)$ usually comes from the theorem of residues, and (b) needs to be verified. □

5 Analytic Continuation of the Integrand

Now from (21a) and (22) for $\xi = 3/2$, [writing $\tau = \xi + \mathrm{i}\eta$; i.e. $\xi = \Re e\,\tau$ from here on] and by use of (24)

$$\begin{aligned} &2^{2s+\tau} D_1(s,\tau) \\ &\quad= \int_{(1/2)}^{*} \Gamma(w) L(s+\tau, s+w) C(w) \cos\tfrac{\pi w}{2}\,dw \quad (\sigma > 1/2,\, \xi = 3/2) \\ &\quad= \int_{(x)}^{*} \Gamma(z-s) L(s+\tau, z) C(z-s) \cos\left[\tfrac{\pi}{2}(z-s)\right]\,dz \quad (x = \sigma + 1/2) \\ &\quad= \int_{N_z}^{*} \text{same} \cdot dz + \Gamma(1-s) R(s+\tau) C(1-s) \sin\tfrac{\pi s}{2}, \;\; (1/4 < \sigma < 3/4) \end{aligned} \tag{28}$$

after shifting the contour over the pole at $z = 1$, justified by (27) and Lemma 4; and the last result holds for $1/4 < \sigma < 3/4$ (at least). Similar, from (21b)

$$2^{2s+\tau} D_2(s,\tau) =: \int_{(-3/4)}^{*} \Gamma(w) D_3(w,s,\tau)\,dw, \quad (\sigma > 1/2,\, \xi = 3/2) \tag{29}$$

where, after substituting $\tilde{w} = z - s$ and with $x = \sigma + 1/2 > 1$, on $u = -3/4$

$$\begin{aligned} &D_3(w,s,\tau) \\ &\quad:= \int_{(x)}^{*} \Gamma(z-s) L(s+\tau+w, z) C(w+z-s) \cos\left[\tfrac{\pi}{2}(w-z+s)\right] dz \\ &\quad= \int_{N_z}^{*} \text{same} \cdot dz + \Gamma(1-s) R(s+\tau+w) C(w+1-s) \sin\left[\tfrac{\pi}{2}(w+s)\right], \end{aligned} \tag{29a}$$

and this represents a holomorphic function of s on $1/4 < \sigma < 3/4$ again. Substituting the last result into (29) yields

$$\begin{aligned} &2^{2s+\tau} D_2(s,\tau) = \\ &\quad \int_{(-3/4)}^{*} \Gamma(w)dw \int_{N_z}^{*} \Gamma(z-s) L(s+\tau+w, z) C(w+z-s) \cos\left[\tfrac{\pi}{2}(w-z+s)\right] dz \\ &\quad + \Gamma(1-s) D_4(s,\tau) \quad \text{(say)}, \quad (1/2 < \sigma < 3/4,\; \xi = 3/2) \end{aligned} \tag{30}$$

where

$$D_4(s,\tau) \ := \ \int_{(-3/4)}^{*} \Gamma(w)R(s+\tau+w)C(w+1-s)\sin\left[\tfrac{\pi}{2}(w+s)\right]dw$$
$$= \ \int_{(1/4)}^{*} same\cdot dw \ - \ R(s+\tau)C(1-s)\sin\tfrac{\pi s}{2}$$
$$\text{for } 1/4<\sigma<3/4 \qquad \text{(30a)}$$

by (10), (24) and Lemma 4, which requires $u<\sigma$ for holomorphy of $C(w+1-s)$.

Now we show that the iterated integral in (30) is abs. conv. at least for $1/2 \le \sigma < 3/4$ and $\xi > 5/4$. We use the following known estimate (see Chandrasekharan[1], p.69) on N_z:

$$\zeta(z) \ll y^{4h(y)/\log h^{-1}(y)}\log y \quad \text{for } x \ge 1-h(y) \ \&\ y>y_0>3. \qquad (31)$$

By (10) and Lemma 3 since now $\sigma+\xi+u \ge \xi-1/4>1$ in (30) and s is fixed,

$$\int_{(-3/4)}|\Gamma(w)|\,dv\int_{N_z}\left|\Gamma(z-s)L(s+\tau+w,z)C(w+z-s)\cos\left[\tfrac{\pi}{2}(w-z+s)\right]\right|dy$$
$$\ll \int_{-\infty}^{\infty}(1+|v|)^{-\frac{5}{4}}dv\int_{-\infty}^{\infty}(1+|y-t|)^{x-\sigma-\frac{1}{2}}|\zeta(z)|\log^3(2+|y|)\cdot(1+|v+y-t|)^{-3}dy$$
$$\ll \int_{-\infty}^{\infty}(1+|v|)^{-\frac{5}{4}}dv\int_{-\infty}^{\infty}(1+|\tilde{y}|)^{-3}d\tilde{y} \ < \ \infty\,, \quad (1/2\le\sigma<3/4,\ \xi>5/4) \qquad (32)$$

since here $x=1-h(y)$ and for $|y|>y_1(t)>y_0$ we have

$$(1+|y-t|)^{x-\sigma-\frac{1}{2}}|\zeta(z)|\log^3(2+|y|)$$
$$\ll \quad (1+|y|)^{-h(y)+4h(y)/\log h^{-1}(y)}\log^4(2+|y|) \quad \ll \quad 1$$

as implied by

$$h(y)=c\cdot[\log(2+|y|)]^{-4/5} \ \ \&\ \ \log h^{-1}(y) > \tfrac{4}{5}\log\log(2+|y|), \quad |y|>y_0.$$

Therefore $D_2(s,\tau)$ is represented in (30) also on $1/2\le\sigma<3/4$ as a holomorphic function of s, when $\xi>5/4$.

Using eqs. (21), (28), (30) and (30a) we obtain finally

$$2^{2s+\tau}D(s,\tau) =$$
$$\int_{N_z}^{*}\Gamma(z-s)L(s+\tau,z)C(z-s)\cos\left[\tfrac{\pi}{2}(z-s)\right]dz$$
$$+ \ \Gamma(1-s)\cdot\int_{(1/4)}^{*}\Gamma(w)R(s+\tau+w)C(w+1-s)\sin\left[\tfrac{\pi}{2}(w+s)\right]dw$$
$$+ \ H_2(s,\tau)\,, \quad (1/2\le\sigma<3/4,\ \xi>5/4) \qquad (33)$$

where from (30)

$$H_2(s,\tau) := \int_{(-3/4)}^{*} \Gamma(w) D_5(w,s,\tau)dw \qquad (1/2 \le \sigma < 3/4,\, \xi > 5/4) \tag{33a}$$

with

$$D_5(w,s,\tau) := \int_{N_z}^{*} \Gamma(z-s)L(s+\tau+w,z)C(w+z-s)\cos\left[\tfrac{\pi}{2}(w-z+s)\right]dz\,; \tag{33b}$$

and we repeat that even the double integral associated with (33a,b) is abs. conv.

6 Analytic Continuation of $T(s,\delta)$

We now consider again $T_2(s,\delta)$, which is represented in (20) for $\sigma > 1$. Inserting $D(s,\tau)$ from (21) and (21a,b) we obtain a representation of $T_2(s,\delta)$ as a holomorphic function of s on $\sigma > 1/2$, since $D_1(s,\tau)$ & $D_2(s,\tau)$ are bounded on $\sigma \ge \sigma_0 > 1/2$ and $\xi = 3/2$ independent of t and η. Then restricting s to $1/2 < \sigma < 3/4$ we can replace $D_1(s,\tau)$ & $D_2(s,\tau)$ according to (28) & (30) and finally their sum $D(s,\tau)$ according to (33), in (20) and obtain

$$\begin{aligned} 2^{2s+1}T_2(s,\delta') &= \int_{(3/2)}^{*} \delta^{-\tau}\Gamma(\tau)2^{2s+\tau}D(s,\tau)\,d\tau \quad (\delta' := \delta/2 > 0) \\ &= F_0(s,\delta) + \Gamma(1-s)F_1(s,\delta) + F_2(s,\delta), \;\; (1/2 \le \sigma < 3/4) \end{aligned} \tag{34}$$

where

$$F_0(s,\delta) := \int_{(3/2)}^{*} \delta^{-\tau}\Gamma(\tau)\,d\tau \int_{N_z}^{*} \Gamma(z-s)L(s+\tau,z)C(z-s)\cos\left[\tfrac{\pi}{2}(z-s)\right]dz, \tag{34a}$$

$$F_1(s,\delta) := \int_{(3/2)}^{*} \delta^{-\tau}\Gamma(\tau)\,d\tau \int_{(1/4)}^{*} \Gamma(w)R(s+\tau+w)C(1+w-s)\sin\left[\tfrac{\pi}{2}(w+s)\right]dw, \tag{34b}$$

$$F_2(s,\delta) := \int_{(3/2)}^{*} \delta^{-\tau}\Gamma(\tau)H_2(s,\tau)\,d\tau, \quad (1/2 \le \sigma < 3/4) \tag{34c}$$

with $H_2(s,\delta)$ from (33a,b). We note that here also the associated double integrals are abs. conv., and so is even the triple integral associated with (34c), as is visible by using (32); thus allowing s to be also on $\sigma = 1/2$.

7 The Limit of $F_1(s,\delta)$ as $\delta \searrow 0$

Next we study the behavior of $T_2(s,\delta')$ as $\delta \searrow 0$. Since $F_0(s,\delta)$ will cancel out later we begin with $F_1(s,\delta)$. Restricting s again to $1/2 < \sigma < 3/4$ we can shift the contour of the inner integral in (34b) to $u = 1/2$; [i.e. keeping it to the left of the pole $w = s$ of $C(1 + w - s)$]. The resulting iterated integral still is abs. conv. and so

$$F_1(s,\delta) \;=\; \int^*_{(3/2)} \delta^{-\tau}\Gamma(\tau)H_1(s,\tau)\,d\tau \quad \text{for } \sigma > 1/2 \text{ with} \tag{35}$$

$$H_1(s,\tau) \;:=\; \int^*_{(1/2)} R(s+\tau+w)\Gamma(w)C(1-s+w)\sin\left[\tfrac{\pi}{2}(s+w)\right]\,dw. \tag{35a}$$

It follows from the definition of $R(z)$ in (24) and by abs. conv. of the integral in (35a) that $H_1(s,\tau)$ is for s on $\sigma > 1/2$ a holomorphic & bounded function of τ on the half-plane $\Re e\,\tau = \xi > 1/2 - \sigma$. Therefore, when $\sigma = 1/2 + 2\epsilon$ say, we can shift the contour in (35) from $\xi = 3/2$ over the pole $\tau = 0$ to $\xi = -\epsilon < 0$ and obtain

$$F_1(s,\delta) = \int^*_{(-\epsilon)} \delta^{-\tau}\Gamma(\tau)H_1(s,\tau)\,d\tau + H_1(s,0), \quad (\sigma = 1/2 + 2\epsilon < 3/4) \tag{35b}$$

and thus

$$F_1(s) := \lim_{\delta\searrow 0} F_1(s,\delta) = H_1(s,0) \quad \text{for } 1/2 < \sigma < 3/4. \tag{36}$$

Now, substituting $w = z - s$ in (35a) and using (24) for the residue at the pole $z = 1$ of $R(z)$, we get after shifting the contour to N_z

$$\begin{aligned} H_1(s,0) &= \int^*_{(x)} R(z)\Gamma(z-s)C(1-2s+z)\sin\tfrac{\pi z}{2}\,dz, \quad x = 1/2+\sigma > 1 \\ &= M''(1,1)\Gamma(1-s)C(2-2s) + G_1(s), \quad (1/2 < \sigma < 3/4) \end{aligned}$$

where

$$G_1(s) := \int^*_{N_z} R(z)\Gamma(z-s)C(1-2s+z)\sin\tfrac{\pi z}{2}\,dz, \quad (1/2 \le \sigma < 3/4) \tag{37}$$

represents a holomorphic function of s even on $1/2 \le \sigma < 3/4$, by Lemma 4; and we recall that $N_z = \{z = 1 - h(y) + \mathrm{i}y,\ -\infty < y < \infty\}$. Hence we have

$$F_1(s) = 4B_2\Gamma(1-s)C(2-2s) + G_1(s), \quad (1/2 \le \sigma < 3/4,\ s \ne 1/2). \tag{38}$$

8 The Limit of $F_2(s,\delta)$ as $\delta \searrow 0$

Next we study $F_2(s,\delta)$ given in (34c) with (33a,b) for $1/2 \le \sigma < 3/4$ and $\delta > 0$, where the associated triple integral conv. abs. We keep s and δ fixed in the following unless specified otherwise. Replacing w by $w - s$ in (33a) and inserting this in (34c) we obtain after an allowed exchange of integrations

$$F_2(s,\delta) = \int^*_{(u)} \Gamma(w-s)\,dw \int^*_{(3/2)} \delta^{-\tau}\Gamma(\tau)D_5(w-s,s,\tau)\,d\tau, \quad u = \sigma - 3/4. \tag{39}$$

To enable working with abs. conv. integrals we use the fact that

$$\lim_{\epsilon \searrow 0} \int_{(u)}^{*} e^{\epsilon w^2} F(w)\, dw \quad \text{exists and equals} \quad \int_{(u)}^{*} F(w)\, dw$$

if the latter integral exists in the sense of (9); i.e. if it conv. abs. So

$$F_2(s,\delta) = \lim_{\epsilon \searrow 0} \int_{(u)}^{*} e^{\epsilon w^2} \Gamma(w-s) K(w,s,\delta)\, dw, \quad u = \sigma - 3/4 \tag{40}$$

with

$$K(w,s,\delta) := \int_{(3/2)}^{*} \delta^{-\tau} \Gamma(\tau) D_6(w,s,\tau)\, d\tau, \quad D_6(w,s,\tau) := D_5(w-s,s,\tau). \tag{40a}$$

From (33b) follows that $D_6(w,s,\tau)$ is for every τ on $\xi = \Re e\, \tau = 3/2$ a holomorphic function of w on the strip: $-1/2 = 1 - \xi < u < 2\sigma$; then so is $K(w,s,\delta)$, [observing the properties of $L(\tau+w,z)$ and $C(w+z-2s)$]. Noting $\left|\cos\left[\frac{\pi}{2}(w-z)\right]\right| \le e^{\frac{\pi}{2}(|v|+|y|)}$ and $\Gamma(z-s) \ll |y|^{x-\sigma-1/2} e^{-\frac{\pi}{2}|y|}$ as $|y| \to \infty$ it follows that $D_6(w,s,\tau) \ll e^{\frac{\pi}{2}|v|}$ and also $K(w,s,\delta) \ll e^{\frac{\pi}{2}|v|}$ as $|v| \to \infty$ in $-1/4 \le \sigma - 3/4 \le u \le u_1 < 2\sigma$ with $u_1 := \sigma + 1/8$, say. Now we can shift the contour of the integral in (40) from $u = \sigma - 3/4$ to the right over the pole $w = s$ of $\Gamma(w-s)$ to $u_1 = \sigma + 1/8$ and obtain

$$\begin{aligned} \int_{(u)}^{*} e^{\epsilon w^2} \Gamma(w-s) K(w,s,\delta)\, dw &= \frac{1}{2\pi \mathrm{i}} \int_{(u_1)} \text{same} \cdot dw - e^{\epsilon s^2} K(s,s,\delta) \ (\text{for } \epsilon \ge 0) \\ &= \int_{(u_1)}^{*} \text{same} \cdot dw - e^{\epsilon s^2} F_0(s,\delta), \quad \text{for } \epsilon > 0 \end{aligned}$$

the last by (40a), (33b) and (34a). Hence by (40)

$$\begin{aligned} F(s,\delta) &:= F_0(s,\delta) + F_2(s,\delta) \\ &= \lim_{\epsilon \searrow 0} \int_{(u_1)}^{*} e^{\epsilon w^2} \Gamma(w-s) K(w,s,\delta)\, dw, \quad u_1 = \sigma + 1/8. \end{aligned} \tag{41}$$

Next substituting $\tau = \tau' - w$, $d\tau = d\tau'$ in (40a) yields (for $u = u_1$, say)

$$K(w,s,\delta) = \int_{(\xi)}^{*} \delta^{w-\tau} \Gamma(\tau - w) D_7(w,s,\tau)\, d\tau, \quad \xi = u + 3/2 \tag{42}$$

where with (33b)

$$\begin{aligned} D_7(w,s,\tau) &:= D_6(w,s,\tau-w) \\ &= \int_{N_z}^{*} L(\tau,z) \Gamma(z-s) C(w+z-2s) \cos\left[\tfrac{\pi}{2}(w-z)\right] dz. \end{aligned} \tag{43}$$

By (22) we can write $D_7(w,s,\tau) =: \zeta_2(\tau) D_8(w,s,\tau)$, where D_8 is a holomorphic function of $\tau = \xi + \mathrm{i}\eta$ on $\xi \ge 1 - h(\eta)$ for every w on $u = u_1 < 2\sigma$, by Lemma

3. Using also the estimate of $M''(\tau,z)$ and (25) we can shift the contour of the integral in (42) from $\xi = u + 3/2$ to the left over the pole $\tau = 1$ of $\zeta_2(\tau)$ to N_τ, and observing that $u_1 = \sigma + 1/8 < 7/8 < 1 - h(0) \le 1 - h(\eta) \le \xi$, [i.e. that $\Gamma(\tau - w)$ for w on $u = u_1$, remains holomorphic resp. τ in the region needed for the shift] we obtain

$$K(w,s,\delta) = \int_{N_\tau}^{*} \delta^{w-\tau}\Gamma(\tau - w)\zeta_2(\tau)D_8(w,s,\tau)\,d\tau + \delta^{w-1}\Gamma(1-w)D_8(w,s,1) \tag{44}$$

for w on $u = u_1$; and from (43) by (24)

$$D_8(w,s,1) = \int_{N_z}^{*} R(z)\Gamma(z-s)C(w+z-2s)\cos\left[\tfrac{\pi}{2}(w-z)\right]\,dz. \tag{45}$$

Denoting the integral in (41) as $J(s,\delta,\epsilon)$ and inserting K from (44) yields

$$\begin{aligned} J(s,\delta,\epsilon) \quad &= \quad \int_{(u_1)}^{*} e^{\epsilon w^2}\Gamma(w-s)K_0(w,s,\delta)\,dw \\ &\quad + \int_{(u_1)}^{*} e^{\epsilon w^2}\delta^{w-1}\Gamma(w-s)\Gamma(1-w)D_8(w,s,1)\,dw \end{aligned}$$

for $\epsilon > 0$ with

$$K_0(w,s,\delta) := \int_{N_\tau}^{*} \delta^{w-\tau}\Gamma(\tau-w)D_7(w,s,\tau)\,d\tau \quad (u < 1 - h(0)) \tag{46}$$

by Lemma 6, since the second integral for J conv. abs.; and even for $\epsilon = 0$, on account of the two Γ-factors. Hence from (41)

$$F(s,\delta) = \lim_{\epsilon \searrow 0} J(s,\delta,\epsilon) = \lim_{\epsilon \searrow 0} J_0(s,\delta,\epsilon) + F_3(s,\delta), \tag{47}$$

where

$$J_0(s,\delta,\epsilon) := \int_{(u)}^{*} e^{\epsilon w^2}\Gamma(w-s)K_0(w,s,\delta)\,dw, \quad u = u_1,\ \epsilon > 0 \tag{48}$$

and

$$F_3(s,\delta) := \int_{(u)}^{*} \delta^{w-1}\Gamma(w-s)\Gamma(1-w)D_8(w,s,1)\,dw, \quad u = \sigma + 1/8. \tag{49}$$

Next upon inserting K_0 from (46) in (48) for $\epsilon > 0$ we have an abs. conv. iterated integral and may exchange integrations, i.e.

$$J_0(s,\delta,\epsilon) = \int_{N_\tau}^{*} d\tau \int_{(u)}^{*} \delta^{w-\tau}\Gamma(\tau-w)e^{\epsilon w^2}\Gamma(w-s)D_7(w,s,\tau)\,dw, \tag{50}$$

where $u = \sigma + 1/8 < 1 - h(0) \le \xi$ for τ on N_τ; [noting that $\left|e^{\epsilon w^2}\right| = e^{\epsilon(u^2 - v^2)}$, and by (43), (22) and Lemma 3 that $D_7(w,s,\tau) \ll e^{\frac{\pi}{2}|v|}|\zeta(\tau)|\log^3(2+|\eta|)$ and using (31) say, for justification]. Finally we may shift the contour of the inner integral in (50) to the right over the pole $w = \tau$ of $\Gamma(\tau - w)$ to $u = 2\sigma \geq 1$, observing that $D_7(w, s, \tau)$ of (43) represents a holomorphic function of w on $u \leq 2\sigma$, since then $z + w - 2s$ remains to the left of the pole $z = 1 + 2s - w$ of $C(z + w - 2s)$ as z traverses N_z. This yields for $\epsilon > 0$

$$J_0(s, \delta, \epsilon) = J_1(s, \delta, \epsilon) + \int_{N_\tau}^{*} e^{\epsilon\tau^2}\Gamma(\tau - s)D_7(\tau, s, \tau)\, d\tau, \quad 1/2 \leq \sigma < 3/4 \quad (51)$$

with

$$J_1(s, \delta, \epsilon) := \int_{N_\tau}^{*} d\tau \int_{(2\sigma)}^{*} \delta^{w-\tau}\Gamma(\tau - w)e^{\epsilon w^2}\Gamma(w - s)D_7(w, s, \tau)\, dw, \qquad (52)$$

since each of these integrals is abs. conv. indeed, by the estimations noted below (50). We now need our

Main Lemma $\lim_{\epsilon\searrow 0} \int_{N_\tau}^{*} e^{\epsilon\tau^2}\Gamma(\tau - s)D_7(\tau, s, \tau)\, d\tau$ *exists for* $1/2 \leq \sigma < 3/4$ *and equals*

$$H(s) := \frac{1}{2\pi \mathrm{i}} \int_{N_\tau} \Gamma(\tau - s)\, d\tau \int_{N_z}^{*} L(\tau, z)\Gamma(z - s)C(\tau + z - 2s) \cos\left[\tfrac{\pi}{2}(w - z)\right] dz \qquad (53)$$

representing a holomorphic function of s *on* $1/2 < \sigma < 3/4$*, which is continuous also on* $1/2 \leq \sigma < 3/4$.

Temporarily deferring the proof of this lemma we use it to infer from (47) and (51) that

$$F(s, \delta) = \lim_{\epsilon\searrow 0} J_1(s, \delta, \epsilon) + H(s) + F_3(s, \delta) \quad \text{for } 1/2 \leq \sigma < 3/4 \ \&\ \delta > 0. \qquad (54)$$

9 The Limit of $T(s, \delta)$ as $\delta \searrow 0$

Observing the definition of $F(s, \delta)$ in (41) we finally obtain from (34)

$$2^{2s+1}T_2(s, \delta') = F(s, \delta) + \Gamma(1-s)F_1(s, \delta), \quad (1/2 \leq \sigma < 3/4,\ \delta' = \delta/2 > 0). \quad (55)$$

For the behavior of $F_1(s, \delta)$ when $\delta \searrow 0$ we already have (36) with (38). Again restricting s to $1/2 < \sigma < 3/4$ and setting $b := \sigma - 1/2$ we have from (52)

$$\lim_{\epsilon\searrow 0} J_1(s, \delta, \epsilon) = \delta^{2b} \lim_{\epsilon\searrow 0} \int_{N_\tau}^{*} d\tau \int_{(0)}^{*} \delta^{1+z-\tau}\Gamma(\tau - w)e^{\epsilon w^2}\Gamma(w - s)D_7(w, s, \tau)\, dz$$

with $w = 2\sigma + z$, (the limit exists by the foregoing) and here $\Re e(1 + z - \tau) = 1 - \xi = h(\eta) > 0$. Since $b > 0$ this yields

$$\lim_{\delta\searrow 0} \lim_{\epsilon\searrow 0} J_1(s, \delta, \epsilon) = 0 \quad \text{for } 1/2 < \sigma < 3/4, \qquad (56)$$

for use in (54). Turning to $F_3(s,\delta)$ given in (49) we can shift the contour of the integral from $u = \sigma + 1/8 < 1$ to the right over the pole $w = 1$ of $\Gamma(1-w)$ to $u = 2\sigma = 1+2b > 1$, since $D_8(w,s,1)$ in (45) is holom. on $u \le 2\sigma$, in particular. So

$$F_3(s,\delta) = \Gamma(1-s)D_8(1,s,1) + \delta^{2b} \int_{(1+2b)}^{*} \delta^{iv}\Gamma(w-s)\Gamma(1-w)D_8(w,s,1)\,dw$$

and then

$$\begin{aligned} F_3(s) &:= \lim_{\delta \searrow 0} F_3(s,\delta) \\ &= \Gamma(1-s)D_8(1,s,1) = \Gamma(1-s)G_1(s) \quad \text{for } 1/2 < \sigma < 3/4, \end{aligned} \tag{57}$$

the last by comparing (45) for $w = 1$ with (37). Now from (54) to (57) follows

$$\begin{aligned} 2^{2s+1}T_2(s,0) &= H(s) + F_3(s) + \Gamma(1-s)F_1(s) \quad (1/2 < \sigma < 3/4) \\ &= 4B_2\Gamma^2(1-s)C(2-2s) + 2\Gamma(1-s)G_1(s) + H(s). \end{aligned} \tag{58}$$

10 Proof of Theorem 1

Returning to $T(s,\delta)$ of (11) we obtain from (19) for $\delta = 0$ and by (58) & (17)

$$\begin{aligned} T(s,0) &= T_2(s,0) - T_0(s,0) + \frac{1}{2}A(s,0) \\ &= 2^{1-2s}B_2\Gamma^2(1-s)C(2-2s) + 2^{-2s}\Gamma(1-s)G_1(s) + 2^{-1-2s}H(s) \\ &\quad -\frac{B_2}{2s-1} - B(2s) + \frac{1}{2}A(s,0), \quad (1/2 < \sigma < 3/4) \end{aligned} \tag{59}$$

and here by Lemma 4 the term containing $C(2-2s)$ has a pole at $s = 1/2$ with principal part $2B_2/(2s-1)$, $G_1(s)$ defined in (37) is holomorphic even on $1/2 \le \sigma < 3/4$, $H(s)$ defined in (53) is holomorphic on $1/2 < \sigma < 3/4$ and continuous on $1/2 \le \sigma < 3/4$, $B(2s)$ defined in (17) is holomorphic on $\sigma \ge 1/2$ and $A(s,0) = 3^{-s}\log^2 3$ from (13). Hence we can rewrite (59) as

$$T(s,0) = \frac{B_2}{2s-1} + F(s), \qquad (1/2 \le \sigma < 3/4) \tag{59'}$$

where $F(s)$ is holomorphic on $1/2 < \sigma < 3/4$ and continuous on $1/2 \le \sigma < 3/4$. Next, by (1) and (12), for $\sigma > 1$ at first,

$$f(s) := T(s/2,0) - T(s) = \sum_{n>3} \Lambda(n-1)\Lambda(n+1)n^{-s}\left[(1-n^{-2})^{-s/2} - 1\right], \tag{60}$$

and this series conv. abs. for $\sigma > -1$ and thus represents a holomorphic function there. Finally (60) and (59') yield

$$T_1(s) = T(s) - \frac{B_2}{s-1} = F(s/2) - f(s) \quad \text{for } 1 \le \sigma < 3/2, \tag{61}$$

and this represents a continuous function of s also on $\sigma = 1$, in particular, thereby completing the proof of Theorem 1.

11 Proof of the Main Result

Theorem 1 supplies the prerequisites for an application of the complex Tauberian theorem of N. Wiener and S. Ikehara to $T(s)$ of (1). Setting

$$T^*(x) := \sum_{3<n\le x} \Lambda(n-1)\Lambda(n+1) \quad (x \ge 4), \quad T^*(x) := 0 \quad (x < 4) \tag{62}$$

we have by partial summation

$$T(s) = s\int_1^\infty T^*(x)x^{-s-1}\,dx = s\int_0^\infty e^{-su}T^*(e^u)\,du \quad \text{for } \sigma > 1,$$

and $T^*(x)$ is non-negative and non-decreasing for increasing $x > 1$. Now the Wiener-Ikehara Theorem [7,p.233] or [2,p.481] and Theorem 1 imply

$$T^*(x) = B_2 x + o(x) \quad \text{as } x \to \infty; \text{ i.e. } \lim_{x\to\infty} x^{-1}T^*(x) = B_2. \tag{63}$$

For $3 < n \le N$ and $N > 5$ we note that $\Lambda(n-1)\Lambda(n+1) \ne 0$ only if $n-1 = p^k$ & $n+1 = q^\ell$ with odd primes p & q and k, ℓ in $\mathbb{Z}^+$, and then $p^k < N$ and $q^\ell \le N+1$ imply $p < N$, $k < \log N$, and $q \le N+1$, $\ell < \log N$. So from (62) for $N > 5$

$$T^*(N) = \sum_{\substack{k,\ell\ge 1 \\ 8\le p^k+q^\ell\le 2N,\, q^\ell-p^k=2}} \sum_{p,q} \log p \log q = \sum_{p<N} \sum_{\substack{q \\ q-p=2}} \log p \log q + R(N), \tag{64}$$

where p & q stand for odd primes and

$$\begin{aligned} R(N) &:= \sum_{\substack{\ell>1 \\ q^\ell\le N+1,\, p=q^\ell-2}} \sum_{p,q} \log p \log q + \sum_{\substack{k>1 \\ \ell\ge 1}} \sum_{\substack{p,\,q \\ p^k<N,\, q^\ell=p^k+2}} \log p \log q \\ &\le (\log N)^2 \left(\sum_{\ell\ge 2} \sum_{q^\ell\le N+1} 1 + \sum_{k\ge 2} \sum_{p^k<N} 1 \right) \ll N^{1/2}(\log N)^2. \end{aligned} \tag{64a}$$

Now (63), (64) and (64a) yield our main result; i.e.

Theorem 2

$$T_2(N) := \sum_{\substack{p<N \\ p\,\&\,p+2 \text{ both prime}}} \log p \log(p+2) = B_2 N + o(N) \quad \textit{as } N \to \infty.$$

12 Proof of the Main Lemma

We have still to prove the Main Lemma (used without proof in the derivation of Theorem 1). To this end we shall prove the existence (i.e. conditional convergence) of the iterated integral in (53) and, setting

$$J(s,\epsilon) := \int_{N_w}^{*} e^{\epsilon w^2}\Gamma(w-s)D_7(w,s,w)\,dw \quad \text{for } 1/2 \le \sigma < 3/4 \text{ \& } \epsilon > 0, \tag{65}$$

that $J(s,\epsilon) \to H(s)$ as $\epsilon \to 0$, and that $H(s)$ has the properties stated under (53). From here on we fix s and ϵ and sometimes omit s from the notation. Now

$$J(s,\epsilon) = \frac{1}{2\pi}\int_0^\infty e^{\epsilon w^2} b(v)\,dv + \frac{1}{2\pi}\int_0^\infty e^{\epsilon \overline{w}^2} b(-v)\,dv =: J_1(\epsilon) + J_2(\epsilon), \quad (\epsilon > 0) \tag{66}$$

where

$$b(v) := \Gamma(w-s)D_7(w,s,w) \ \text{ on } w = w(v) := 1 - h(v) + \mathrm{i}v; \text{ (i.e. } w \text{ on } N_w) \tag{66a}$$

$h(-v) = h(v)$ as in (23), and D_7 is given in (43) with τ replaced by w; and so, using also (22)

$$\begin{aligned} D_7(w,s,w) &= \zeta_2(w)D(w), \\ D(w) &:= \int_{N_z}^{*} M''(w,z)\zeta_2(z)\Gamma(z-s)C(w+z-2s)\cos\left[\tfrac{\pi}{2}(w-z)\right] dz \end{aligned} \tag{66b}$$

1. At first we assume (and prove later) that

$$\int_0^\infty b(v)\,dv =: 2\pi J_1(0) \ \text{ and } \ \int_0^\infty b(-v)\,dv =: 2\pi J_2(0) \ \text{ both exist.} \tag{A*}$$

Setting

$$E(v,\epsilon) := e^{\epsilon w^2} = e^{-\epsilon(v^2-u^2-2\mathrm{i}uv)} \ \text{ and } \ B(x) := \int_0^x b(v)\,dv \ \text{ for } w = w(v)$$

we have

$$\begin{aligned} \int_0^x E(v,\epsilon)b(v)\,dv &= E(x,\epsilon)B(x) - \int_0^x E'(v,\epsilon)B(v)\,dv, \\ \int_0^x E'(v,\epsilon)\,dv &= E(x,\epsilon) - E(0,\epsilon); \end{aligned}$$

and so

$$\begin{aligned} &\int_0^x e^{\epsilon w^2} b(v)\,dv - E(0,\epsilon)B(\infty) = \\ &\qquad E(x,\epsilon)[B(x) - B(\infty)] + \int_0^x E'(v,\epsilon)[B(\infty) - B(v)]\,dv \end{aligned} \tag{67}$$

with $B(\infty) = 2\pi J_1(0)$. Also, for $v \geq 0$, $0 < \epsilon < 1$ and $0 < a < x$

$$|E(v,\epsilon)| \leq e^{-\epsilon v^2 + 1} \leq e \ \text{ and}$$

$$\int_a^x |E'(v,\epsilon)|\,dv \leq 2e\int_0^\infty \epsilon e^{-\epsilon v^2}(1+v)(1-h'(v))\,dv = O(1) \ \text{ as } \epsilon \searrow 0.$$

Therefore each term on the right hand side of (67) converges as $x \to \infty$, and even uniformly for ϵ in $[0,1]$. Hence (67) for $x = \infty$ yields

$$\lim_{\epsilon \searrow 0} \int_0^\infty e^{\epsilon w^2} b(v)\, dv = E(0,0)B(\infty) = 2\pi J_1(0);$$

i.e. $J_1(\epsilon) \to J_1(0)$, and likewise $J_2(\epsilon) \to J_2(0)$ as $\epsilon \to 0$, and so by (66) and (66a)

$$\lim_{\epsilon \searrow 0} J(s,\epsilon) = \frac{1}{2\pi} \int_{-\infty}^\infty b(v)\, dv = \frac{1}{2\pi \mathrm{i}} \int_{N_w} \Gamma(w-s) D_7(w,s,w)\, dw = H(s), \quad (68)$$

the last by (43) and (53) with τ replaced by w. Thus, the assumption A* [which is equivalent to assuming existence of the integral $H(s)$], implies with (65) that $J(s,\epsilon) \to H(s)$ as $\epsilon \to 0$.

2. We turn to proving the validity of A*. For this by (66a,b) we need a more detailed description of $D(w)$. From (7a,b,c)

$$M''(w,z) = f''(w)B_0 + f'(w)B_1 + f(w)B_2, \quad f(w) := \zeta_0^{-1}(w), \; B_k = B_k(w,z), \quad (69)$$

where

$$B_0 := Pf(z), \; B_1 := 2Pf'(z) + 2P'f(z), \; B_2 := Pf''(z) + 2P'f'(z) + P''f(z) \quad (69a)$$

with

$$P = P(w,z) \text{ from (6)}, \quad P' := \left(\frac{\partial}{\partial w} + \frac{\partial}{\partial z}\right) P(w,z) \text{ and}$$
$$P'' := \left(\frac{\partial}{\partial w} + \frac{\partial}{\partial z}\right)^2 P(w,z). \quad (69b)$$

Setting

$$\begin{aligned} Q &= Q(w,z;s) \\ &:= \zeta_2(z)\Gamma(z-s)C(w+z-2s)\cos\left[\tfrac{\pi}{2}(w-z)\right], \quad (1/2 \le \sigma < 3/4) \quad (70) \end{aligned}$$

we get from (66b) and (69)

$$\begin{aligned} D(w) &= f(w) \int_{N_z}^* B_2 Q\, dz + f'(w) \int_{N_z}^* B_1 Q\, dz + f''(w) \int_{N_z}^* B_0 Q\, dz \quad \text{on } N_w \\ &=: D_0(w) + D_1(w) + D_2(w) \text{ with } D_k(w) := f^{(k)}(w) J_{2-k}(w), \text{ say.} \quad (71) \end{aligned}$$

To prove A* it suffices to show convergence of the integrals

$$J_k^* :\to \int_{N_w} \Gamma(w-s)\zeta_2(w) D_k(w)\, dw \quad (k = 0,1,2; \; 1/2 \le \sigma < 3/4), \quad (71a)$$

since this by (66a,b) and (71) implies the existence of $H(s)$. To estimate $D_k(w)$ on N_w we observe that (as in the proof of Lemma 3) P, P' and P'' of (69b) are abs. bounded on $N_w \times N_z$, and that

$$f(z),\ f'(z) \text{ and } f''(z) \ll h^{-3}(y) \text{ on } N_z \text{ and so } B_k \ll h^{-3}(y)$$

by (69a). Then by (71), using (31) and (10) to estimate Q of (70) on $N_w \times N_z$,

$$D_k(w) \ll h^{-3}(v) \int_{N_z} h^{-3}(y)|Q(w,z;s)|\,dy \ll e^{\frac{\pi}{2}|v|}h^{-3}(v) \quad \text{for } |v| \gg 1 \tag{71b}$$

similar to the derivation of (32), for $k = 0, 1, 2$. Then, again by (10) and (31),

$$\begin{aligned}\Gamma(w-s)\zeta_2(w)D_k(w) &\ll (1+|v-t|)^{u-\sigma-1/2}(2+|v|)^{4h(v)/\log h^{-1}(v)}\log^4(2+|v|) \\ &\ll (2+|v|)^{-h(v)[1-5/\log\log(2+|v|)]}e^{4\log\log(2+|v|)} \\ &\ll e^{-c_1[\log(2+|v|)]^{1/5}} \ll [\log(2+|v|)]^{-4}, \quad (\text{on } N_w)\end{aligned} \tag{72}$$

say, for $|v| \gg 1$ (and fixed s) and $k = 0, 1, 2$. This shows a rather weak decay of the integrand of J_k^* and indicates that the oscillatory behavior of the integrand also needs to be exploited for the convergence proof of J_k^*.

3. It is known that (for fixed $s = \sigma + it$ and $1/4 < \sigma < 3/4 < u < 5/4$, say)

$$\arg\Gamma(w-s) = (v-t)\log v - v + \frac{\pi}{2}(u-\sigma-1/2) + O(v^{-1}) \quad \text{as } v \gg 1. \tag{73}$$

Next we observe that

$$\begin{aligned}&\zeta_2(w)f(w) = 2^w, \quad \zeta_2(w)f'(w) = -2^w\frac{\zeta_0'}{\zeta_0}(w) = -2^w\left(\frac{\log 2}{2^w-1} + \frac{\zeta'}{\zeta}(w)\right), \\ &\zeta_2(w)f''(w) = 2^w\left[\frac{\zeta_0'}{\zeta_0}(w)\right]^2 - 2^w\left[\frac{\zeta_0'}{\zeta_0}(w)\right]' \quad \& \quad \frac{\zeta'}{\zeta}(w) = Z(w) - S(w),\end{aligned} \tag{74}$$

where

$$S(w) := \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(1+\frac{w}{2}\right) - b + \frac{1}{w-1} \quad \& \quad Z(w) := \sum_\rho\left(\frac{1}{w-\rho} + \frac{1}{\rho}\right); \tag{74a}$$

(cf. [4], p. 31). We shall use (74) and (71) to decompose the integrals of (71a) for $k = 1$ & 2; since it seems difficult to derive useful estimates of $\psi_k(v) := \arg f^{(k)}(w)$ and $\psi_k'(v)$ for w on N_w; while J_0^* can be treated directly. But first we consider

$$S_0(w) := S(w) - \frac{\log 2}{2^w-1}; \quad \text{so that } \frac{\zeta_0'}{\zeta_0}(w) = Z(w) - S_0(w). \tag{75}$$

Now

$$\frac{\Gamma'}{\Gamma}(z) = \operatorname{Log} z + O(|z|^{-1}) \quad \text{as } |z| \to \infty \text{ in } x \geq 0 \tag{76}$$

yields

$$2S_0(w) = \log|2+w| + \mathrm{i}\arg(2+w) + O(1) \quad \text{as } |v|\to\infty \ \&\ u > 1/2; \qquad (77)$$

i.e.

$$\Re e S_0(w) > \frac{1}{5}\log(4+v^2) \quad \text{and} \quad \arg S_0(w) = O(1) \quad \text{for } |v| \gg 1.$$

Also, since we may differentiate in (76), $S_0'(w) = O(1)$ here, by (75). Now if, for example, $\log f(z) = U + \mathrm{i}V$, $V = \arg f(z)$, (where $f(z)$ is holomorphic and $\log f(z)$ is defined) then for the partial derivatives

$$U_x + \mathrm{i}V_x = f'(z)/f(z) \quad \text{and} \quad V_y = U_x; \quad \text{so} \quad |V_x|,\, |V_y| \le |f'(z)/f(z)|.$$

Applying this with $f = S_0$ and taking $w = 1 - h(v) + \mathrm{i}v = w(v)$ we obtain

$$\alpha_0(v) := \arg S_0(w) = O(1) \quad \text{and} \quad \alpha_0'(v) = o(1) \quad \text{on } N_w \text{ as } |v|\to\infty. \qquad (78)$$

Next we consider $Z(w)$ in the region $u > 1 - 3h(v)$, where it is holomorphic,

$$Z(w) = S(w) + \frac{\zeta'}{\zeta}(w) \ll \log v + h^{-1}(v) \ll \log v \quad \text{for } v > v_0 > 20, \qquad (79)$$

(cf. the proof of Lemma 3) and $Z(\overline{w}) = \overline{Z(w)}$. Also, with $\rho = \beta + \mathrm{i}\gamma$

$$\Re e Z(w) = \sum_\rho \left(\frac{u-\beta}{|w-\rho|^2} + \frac{\beta}{|\rho|^2}\right) \ge \sum_\rho \frac{\beta}{|\rho|^2} =: b_1 > 0 \quad \text{for } u \ge 1. \qquad (80)$$

Using that here $\beta < 1 - 3h(\gamma)$ for the zeros ρ of $\zeta(w)$ we remark that

$14 < |\gamma| \le v^a$ with $a = 3/2$ implies $\log(2+|\gamma|) < a\log(2+v)$;
i.e. $h(v) < a^{4/5}h(\gamma) < ah(\gamma)$ and $1 - 3h(\gamma) < 1 - 2h(v)$.

Thus, if $u \ge 1 - 2h(v)$ and $|\gamma| \le v^a$, then $u - \beta > 0$ and hence

$$\sum_\rho \frac{u-\beta}{|w-\rho|^2} > \sum_{|\gamma|>v^a} \frac{u-\beta}{|w-\rho|^2} > \sum_{|\gamma|>v^a} \frac{-2h(v)}{(v-\gamma)^2}, \quad (v > v_1,\ a = 3/2). \qquad (81)$$

Now

$$\begin{aligned}
\sum_{\gamma>v^a}(v-\gamma)^{-2} &= \sum_{n=0}^{\infty} \sum_{v^a+n<\gamma\le v^a+n+1} (\gamma-v)^{-2} \le A\sum_{n=0}^{\infty}(v^a+n-v)^{-2}\log(v^a+n) \\
&\le A\left[(v^a-v)^{-2}\log v^a + \int_0^\infty (v^a-v+x)^{-2}\log(v^a+x)\,dx\right] \\
&\le 3A(v^a-v)^{-1}\log v^a < 6Av^{-a}\log v,
\end{aligned}$$

(cf. [4], p. 184, say) and similarly

$$\sum_{\gamma<-v^a}(v-\gamma)^{-2}=\sum_{-\gamma>v^a}(v-\gamma)^{-2}=\sum_{\gamma>v^a}(\gamma+v)^{-2}<5Av^{-a}\log v.$$

Hence with (80) and (81)

$$\Re e Z(w)\geq b_1-2h(v)\sum_{|\gamma|>v^a}(v-\gamma)^{-2}>b_1-22Ah(v)v^{-a}\log v\geq b_2>0$$
$$\text{for } u\geq 1-2h(v) \text{ and } v\geq v_2, \text{ say.} \tag{82}$$

For $T>v_2+1$ let $w_1=1-h(T)+\mathrm{i}T=w(T)$ on N_w. Then by (82)

$$|Z(w_1)|>b_2 \quad\text{and}\quad Z(w)\neq 0 \quad\text{in}\quad |w-w_1|<\frac{1}{2}h(T),$$

also $Z(w)$ is holomorphic there and by (79) (with a generic constant A)

$$|Z(w)|<A\log T;\quad\text{hence}\quad |Z'(w_1)/Z(w_1)|<4h^{-1}(T)\log\left(\frac{A}{b_2}\log T\right)$$

by a standard lemma (see Prachar [3], p. 383, Satz 4.3). Defining

$$\alpha(v):=\arg Z(w)\quad\text{for } w=w(v),\ v\geq 0 \tag{83a}$$

by continuous variation along N_w (taking possible zeros of $Z(w)$ for $0<v<v_2$ into account as usual) we obtain by (82) and the above (replacing T by v)

$$\alpha(v)=O(1)\quad\text{and}\quad \alpha'(v)\ll h^{-1}(v)\log\log v\quad\text{as } v\to\infty. \tag{83b}$$

4. Using (71), (74) and (75) we obtain formally (i.e. disregarding convergence) the following representations of the three integrals in (71a)

$$J_0^* :\to \int_{N_w}\Gamma(w-s)2^w J_2(w)\,dw,$$
$$J_1^* :\to \int_{N_w}\Gamma(w-s)2^w S_0(w)J_1(w)\,dw-\int_{N_w}\Gamma(w-s)2^w Z(w)J_1(w)\,dw, \tag{84a}$$
$$J_2^* :\to \int_{N_w}\Gamma(w-s)2^w S_0^2(w)J_0(w)\,dw-2\int_{N_w}\Gamma(w-s)2^w S_0(w)Z(w)J_0(w)\,dw$$
$$+\int_{N_w}\Gamma(w-s)2^w Z^2(w)J_0(w)\,dw-J_3^*\quad\text{(say)}, \tag{84b}$$

where

$$J_3^* :\to \int_{N_w}\Gamma(w-s)2^w\left[\frac{\zeta_0'}{\zeta_0}(w)\right]' J_0(w)dw\to-\int_{N_w}[\Gamma(w-s)2^w J_0(w)]'\,\frac{\zeta_0'}{\zeta_0}(w)dw$$
$$\to\int_{N_w}[\Gamma(w-s)2^w J_0(w)]'\,S_0(w)dw-\int_{N_w}[\Gamma(w-s)2^w J_0(w)]'\,Z(w)dw, \tag{84c}$$

noting that

$$\lim_{w=w(v)\to\pm i\infty}\Gamma(w-s)2^w J_0(w)\zeta_0'(w)/\zeta_0(w)=0, \quad \text{by (72).}$$

To show the actual existence of J_k^*, $(k=0,1,2)$ it suffices to prove convergence of every integral appearing in (84a,b,c). For this we still need to study the $J_k(w)$ defined in (71) using (69a,b); where for $P(w,z)$ we now use

$$P = P(w,z) = \sum_n{}^{\circ} \mu(n) g(n,w) g(n,z),$$
$$g(n,z) := \prod_{p|n}(p^z-1)^{-1}, \quad (u, x>0,\ u+x>1) \tag{85}$$

from (5). Then

$$|g(n,z)| = n^{-x}\prod_{p|n}|1-p^{-z}|^{-1} \le n^{-x}\prod_{p|n}(1-p^{-x})^{-1} \le n^{-x}\prod_{p|n}(1-3^{-x})^{-1} = n^{-x}b^r$$

where $b=(1-3^{-x})^{-1}>1$ and r is the number of primes dividing n. The Prime Number Theorem yields $r<c\log m/\log\log m$, if m is the product of the first r primes. Thus $m\le n$ and for $x\ge x_0>0$

$$|g(n,z)| < n^{-x}b^{c\log n/\log\log n} = n^{-x+c\log b/\log\log n} \ll n^{-x+o(1)} \quad \text{as } n\to\infty. \tag{85a}$$

This implies that the series in (85) conv. abs. & uniformly resp. (v,y) in $\mathbb{R}^2$, if $u\ge 1/2$ & $x\ge 3/4$ say, (where it is used in the following) and that P' and P'' of (69b) can be similarly represented by termwise differentiating the series for P.

We obtain from (71), (69a) and (85) by allowed termwise integration

$$J_0(w) = \int_{N_z}^{*} B_0 Q\,dz = \int_{N_z}^{*} P(w,z)f(z)Q\,dz = \sum_n{}^{*}\mu(n)g(n,w)q_0(n,w;s) \tag{86}$$

with

$$q_0(n,w;s) := \int_{N_z}^{*} g(n,z)f(z)Q(w,z;s)\,dz \quad \text{on } N_w \tag{86a}$$

and with Q from (70), observing (85a) and that $7/8<x<1$ on N_z. Also from (85) $g(1,z)=1$, $g'(1,z)=g''(1,z)=0$, and for odd $n>1$

$$g'(n,z) = -g(n,z)\sum_{p|n}\frac{\log p}{1-p^{-z}},$$
$$g''(n,z) = g(n,z)\left[\left(\sum_{p|n}\frac{\log p}{1-p^{-z}}\right)^2 + \sum_{p|n}\frac{p^{-z}\log^2 p}{(1-p^{-z})^2}\right]; \tag{87}$$

i.e. for $x \geq x_0 > 0$ and $n \gg 1$, as in the derivation of (85a)

$$\begin{aligned} |g'(n,z)| &\leq n^{-x} b^{r+1} \log n \ \ll \ n^{-x+o(1)}, \\ |g''(n,z)| &\leq n^{-x} b^{r+2} [1+3^{-x}] \log^2 n \ \ll \ n^{-x+o(1)}. \end{aligned} \tag{87a}$$

From (71) and (69a) also

$$\begin{aligned} J_1(w) &= 2 \int_{N_z}^{*} (Pf'(z) + P'f(z)) Q \, dz, \\ J_2(w) &= \int_{N_z}^{*} (Pf''(z) + 2P'f'(z) + P''f(z)) Q \, dz; \end{aligned}$$

and with (69b) and (85) we obtain by termwise integration

$$\begin{aligned} J_k(w) = e_k {\sum_n}^{\circ} \mu(n) \sum_{j=0}^{k} \binom{k}{j} g^{(j)}(n,w) q_{k-j}(n,w;s), \\ e_k := \binom{2}{k}, \ (k = 0,1,2) \end{aligned} \tag{88}$$

with

$$q_k(n,w;s) := \int_{N_z}^{*} [g(n,z)f(z)]^{(k)} Q(w,z;s) \, dz \quad \text{on } N_w, \ (1/2 \leq \sigma < 3/4); \tag{88a}$$

justified by (85a) and (87a) for $1/2 \leq u \leq 1$, say; and including (86). Here we insert Q from (70), split the cosine and obtain with $\tilde{C}(w)$ from Lemma 4

$$q_k(n,w;s) = \pi^{2s-w} \left[e^{\frac{i\pi w}{2}} q_k^{+}(n,w;s) + e^{-\frac{i\pi w}{2}} q_k^{-}(n,w;s) \right] \tag{89}$$

with

$$q_k^{\pm}(n,w;s) := \frac{1}{2} \int_{N_z}^{*} \pi^{-z} [g(n,z)f(z)]^{(k)} \zeta_2(z) e^{\mp \frac{i\pi z}{2}} \Gamma(z-s) \tilde{C}(z+w-2s) \, dz, \tag{89a}$$

and correspondingly from (88)

$$J_k(w) = \pi^{2s-w} \left[e^{\frac{i\pi w}{2}} J_k^{+}(w) + e^{-\frac{i\pi w}{2}} J_k^{-}(w) \right] \quad \text{on } N_w, \ (k = 0,1,2) \tag{90}$$

with

$$J_k^{\pm}(w) := e_k \sum_{j=0}^{k} \binom{k}{j} {\sum_n}^{\circ} \mu(n) g^{(j)}(n,w) q_{k-j}^{\pm}(n,w;s), \quad (1/2 \leq \sigma < 3/4). \tag{90a}$$

We obtain from (89a) by (85a), (87a), (74) and Lemma 4

$$\begin{aligned} q_k^{\pm}(n,w;s) &\ll n^{-7/8} \int_{-\infty}^{\infty} h^{-k}(y)(1+|y-t|)^{-h(y)}(1+|y+v-2t|)^{-2} h^{-1}(y) \, dy \\ &\ll n^{-7/8} \int_{\infty}^{\infty} (1+|y|)^{-2} \, dy \ll n^{-7/8}, \quad \text{unif. on } 1/2 \leq u \leq 1; \end{aligned} \tag{91}$$

and from the form of $\tilde{C}'(w)$ follows that also

$$[q_0^{\pm}(n,w;s)]' \ll n^{-7/8}, \quad \text{unif. on } 1/2 \le u \le 1. \tag{91a}$$

There from (90a) and (87a) for $k = 0, 1, 2$

$$J_k^{\pm}(w) \ \& \ [J_0^{\pm}(w)]' \ll \sum_k{}^{\circ} n^{-u+\epsilon-7/8} \ll A \quad \text{on } N_w, \tag{92}$$

the respective series converging abs. & unif. there, since $7/8 < u < 1$ on N_w.

5. To circumvent the difficulty of deriving useful estimates for $\arg J_k^{\pm}(w)$ and its derivative along N_w, we plan to use the series of (90a) in (90) and (84a,b) and prove the existence of the infinite integrals there by justifying termwise integration. This requires besides (85a), (87a) and (91) only a determination of $\arg g^{(j)}(n,w)$ & $\arg q_j^{\pm}(n,w;s)$, $(j = 0, 1, 2)$ and their derivatives on N_w.

Let again $n \ge 3$ be odd & squarefree and $w = w(v) = 1 - h(v) + \mathrm{i}v$, $(v \gtrless 0)$ on N_w. Then $u > 7/8$; and by (85) for the principal branch of the logarithms

$$\log g(n,w) + w \log n = -\sum_{p|n} \log(1 - p^{-w}) \ll r \log b_0, \quad (r := \sum_{p|n} 1)$$

uniformly on $u \ge 3/4$, where $r < c \log n / \log\log n$ [as for (85a)] and

$$b_0 = (1 - 3^{-3/4})^{-1} < 2;$$

since

$$|\log(1 - p^{-w})| \le \sum_{k=1}^{\infty} \frac{1}{k} p^{-ku} = -\log(1 - p^{-u}) \le \log(1 - 3^{-u})^{-1} \le \log b_0.$$

Also

$$\begin{aligned} |g'(n,w)/g(n,w) + \log n| \quad &= \quad \left| \sum_{p|n} \frac{p^{-w} \log p}{1 - p^{-w}} \right| \le b_0 \sum_{p|n} p^{-u} \log p \\ &\ll \quad \sum_{p|n} p^{-1/2} < \sum_{j=1}^{r} j^{-1/2} < 2r^{1/2} \quad \text{on } u > 3/4. \end{aligned}$$

Hence we obtain for

$$\beta(v) = \beta(n,v) := \arg g(n,w) \quad \text{on } N_w \text{ with } \beta(n,0) := 0 \tag{93}$$

that

$$|\beta(v) + v \log n| < r \quad \& \quad |\beta'(v) + \log n| < 2r^{1/2}, \quad r < c \log n / \log\log n, \quad (n \gg 1). \tag{93a}$$

Next by (87) for $j = 1, 2$

$$\beta_j(v) := \arg\left[(-1)^j g^{(j)}(n, w)\right] = \beta(v) + \arg S_j(n, w) \quad \text{on } w = w(v) \tag{94}$$

with

$$S_1(n, w) := \sum_{p|n} \frac{\log p}{1 - p^{-w}}, \quad S_2(n, w) := S_1^2(n, w) - S_1'(n, w). \tag{94a}$$

For $v = 0$ we have $S_j(n, u) > 0$ and $\arg S_j(n, u) = 0$; $(u > 0)$. Next

$$|\arg(1 - p^{-w})| \leq |\log(1 - p^{-w})| \leq \log b_0 < 0.6 < \pi/5, \quad (u > 3/4)$$

and

$$\Re e(1 - p^{-w})^{-1} \geq |1 - p^{-w}|^{-1} \cos\frac{\pi}{5} \geq \frac{4}{5}(1 + p^{-u})^{-1} \geq \frac{4}{5}(1 + 3^{-3/4})^{-1} > 1/2,$$

hence

$$|\arg S_1(n, w)| < \pi/5 \quad \text{and} \quad |S_1(n, w)| \geq \Re e S_1(n, w) > \frac{1}{2}\log n \quad \text{on } u > 3/4. \tag{95a}$$

Also

$$|S_1'(n, w)| < b_0^2 \sum_{p|n} p^{-u} \log^2 p \ll \sum_{p|n} p^{-a} < \sum_{j=1}^{r} p_j^{-a} \ll r^{1-a} < \frac{1}{2}(\log n)^{1/2} \tag{95b}$$

with $a = 1/2 < u - 1/4$ (say) and r as before. Thus $|S_1'(n, w)/S_1(n, w)| < (\log n)^{-1/2}$ and by (94)

$$|\beta_1(v) - \beta(v)| < \pi/5 \;\;\&\;\; |\beta_1'(v) - \beta'(v)| < (\log n)^{-1/2} \;\; \text{for } n \gg 1 \;\&\; v \text{ in } \mathbb{R}. \tag{96}$$

Next, by (95a)

$$|\arg S_1^2(n, w)| < 2\pi/5, \quad \Re e S_1^2(n, w) > |S_1^2(n, w)| \cos\frac{2\pi}{5} > \frac{1}{12}\log^2 n$$

and so by (94a) & (95b) for $n \gg 1$

$$|S_2(n, w)| \geq \Re e S_2(n, w) > \frac{1}{12}\log^2 n - (\log n)^{1/2} > \frac{1}{13}\log^2 n. \tag{97}$$

Also, similar to (95b)

$$S_1''(n, w) = \sum_{p|n} \frac{1 + p^{-w}}{(1 - p^{-w})^3} p^{-w} \log^3 p \ll 2b_0^3 \sum_{p|n} p^{-u} \log^3 p \ll r^{1/2} < (\log n)^{1/2} \tag{98}$$

for $n \gg 1$ & $u > 3/4$. Now from (94a)

$$\begin{aligned} |S_2'(n,w)| &\le 2|S_1(n,w)S_1'(n,w)| + |S_1''(n,w)| \\ &< 2b_0 \log n \cdot r^{1/2} + (\log n)^{1/2} < 5r^{1/2}\log n \end{aligned}$$

by (95b) & (98), and then by (97) $|S_2'(n,w)/S_2(n,w)| < (\log n)^{-1/2}$ for $n \gg 1$ & $u > 3/4$, since $r \ll \log n/\log\log n$. Finally by (94) and (97)

$$|\beta_2(v) - \beta(v)| < \pi/2 \ \ \& \ \ |\beta_2'(v) - \beta'(v)| < (\log n)^{-1/2} \ \text{ for } n \gg 1 \ \& \ v \text{ in } \mathbb{R}. \quad (99)$$

6. Turning to the functions in (89a) we write generically

$$q(w) := q_k^{\pm}(n,w;s) = \int_{N_z}^{*} Q(z)\tilde{C}(z+w-2s)\,dz \ \text{ and } \ \tilde{q}(v) := q(w(v)) \quad (100)$$

with

$$Q(z) := Q_k^{\pm}(n,z;s) := \frac{1}{2}\pi^{-z}[g(n,z)f(z)]^{(k)}\zeta_2(z)e^{\mp\frac{\mathrm{i}\pi z}{2}}\Gamma(z-s), \quad (100a)$$

where s is in $1/2 \le \sigma < 3/4$ as always. By (91) we already have $\tilde{q}(v) = O(1)$ as $|v| \to \infty$. To study $\arg\tilde{q}(v)$ we consider

$$\tilde{q}(v) - q(u) = \int_{N_z}^{*} Q(z)[\tilde{C}(z+w-2s) - \tilde{C}(z+u-2s)]\,dz, \ \ u = 1-h(v). \quad (101)$$

Here by Lemma 4

$$[\,-\,] = 2\mathrm{i}v[(1-z-w+2s)(1-z-u+2s)]^{-1} + \mathrm{i}v\sum_{n=1}^{\infty}\frac{(\mathrm{i}\pi)^{2n}}{(2n)!}[(2n+\tilde{z}-w)(2n+\tilde{z}-u)]^{-1} \quad (101a)$$

with $\tilde{z} = 1-z+2s$ for shorter notation. Now $|Q(z)| < A\log^{-4}(3+|y|)$ on N_z, certainly, and integrating termwise (justified by abs. conv.) we get

$$\tilde{q}(v) = q(u) + 2\mathrm{i}vA_0(v) + \mathrm{i}v\sum_{n=1}^{\infty}\frac{(\mathrm{i}\pi)^{2n}}{(2n)!}A_n(v), \quad (102)$$

where

$$A_n(v) := \int_{N_z}^{*} Q(z)[(2n+\tilde{z}-w)(2n+\tilde{z}-u)]^{-1}\,dz \quad \text{on } N_w\,. \quad (102a)$$

Since $\Re e(\tilde{z}-u) = \Re e(\tilde{z}-w) = 1-x+2\sigma-u \ge h(y)+h(v) > 0$ on N_z for every $w = w(v)$ and s in $1/2 \le \sigma < 3/4$, we find that

$$E_n(v) := \int_{N_z}^{*} [(2n+\tilde{z}-w)(2n+\tilde{z}-u)]^{-1}\,dz = 0 \quad \text{for } n \ge 0 \ \& \ w = w(v).$$

Therefore with any constant K

$$A_n(v) = KE_n(v) + A_n(v) = \int_{N_z}^{*} [K+Q(z)][(2n+\tilde{z}-w)(2n+\tilde{z}-u)]^{-1}\,dz; \quad (102b)$$

and for instance

$$A_0(v) = \frac{1}{2\pi}\int_{-\infty}^{\infty}[K+Q(z)]\frac{e^{\mathrm{i}[\omega_1(y,v)+\omega_0(y,v)]}}{|(1-z-w+2s)(1-z-u+2s)|}\,dy, \quad z = 1-h(y)+\mathrm{i}y \qquad (103)$$

where

$$\begin{aligned}&\omega_1(y,v) := \arctan[(y+v-2t)/N] \ \ \& \ \ \omega_0(y,v) := \arctan[(y-2t)/N]\\ &\text{with } N := 2\sigma-1+h(y)+h(v) > 0 \text{ are both in } (-\pi/2,\pi/2) \qquad (103a)\end{aligned}$$

for every v in $\mathbb{R}$. Taking $K = 8A$ (say), so that $K^{-1}|Q(z)| < 1/8$ in (102b) and (103), yields $|\arg A_n(v)| < \frac{5}{4}\pi$ (amply) for every $n \geq 0$ and v in $\mathbb{R}$. Also, each integral in (102a) represents a holomorphic function of w on N_w, and

$$A_n(v) \ll \int_{-\infty}^{\infty}[(2n+|y+v|)(2n+|y|)]^{-1}\,dy < \frac{2}{n} \ \text{ and } \ A_n'(v) \ll n^{-2}, \quad (n>0)$$

uniformly on N_w. Thus the series in (102) conv. abs. and unif. on $\mathbb{R}$ and so does the termwise differentiated series. Setting

$$\psi(v) := \arg\tilde{q}(v) \ \text{ for } \ v \gtrless 0 \ \text{ with } \ -\pi < \arg\tilde{q}(0) \leq \pi, \qquad (104)$$

we obtain from (102) and the foregoing (by geometric addition of the vectors $(-1)^n\frac{\pi^{2n}}{(2n)!}A_n(v)$ in $\mathbb{C}$, say) that

$$\psi(v) = O(1) \ \text{ and } \ \psi'(v) = O(1) \ \text{ as } |v| \to \infty; \qquad (104a)$$

the last also since $\tilde{q}'(v)/\tilde{q}(v) = O(1)$ for $v \to \pm\infty$, as implied by differentiating in (102). We still need to consider [for $J_0'(w)$ in (84c) by (90a)]

$$\psi_1(v) := \arg[q_0^{\pm}(n,w;s)]' \ \text{ on } N_w \text{ with } -\pi < \psi_1(0) \leq \pi. \qquad (105)$$

This amounts to dealing with

$$q'(w) = \int_{N_z}^{*} Q(z)\tilde{C}'(z+w-2s)\,dz \ \text{ on } N_w \qquad (100b)$$

in the notation of (100) & (100a) for $k = 0$. Calculating $q'(w) - q'(u)$ (f.i. by differentiating in (101a) by z) we obtain analogues of (102) and (102b) etc., and again these lead to

$$\psi_1(v) = O(1) \ \text{ and } \ \psi_1'(v) = O(1) \ \text{ as } |v| \to \infty. \qquad (105a)$$

We note that the estimates in (104a) & (105a) are uniform regarding the parameter n occurring in (100) & (105).

7. We return to proving convergence of the infinite integrals in (84a,b,c). Since the proof is substantially the same for all these integrals, we consider just the second integral in (84b) as a model case and write

$$J_{22}^{*} :\to \int_{N_w} \Gamma(w-s)2^w S_0(w)Z(w)J_0(w)\,dw =: \int_{N_w} F_{22}(w)J_0(w)\,dw \quad \text{say,} \tag{106a}$$

and

$$-\,\mathrm{i}J_{22}^{*} = \lim_{T\to\infty}\int_0^T F_{22}(w)J_0(w)\,dv + \lim_{T\to\infty}\int_0^T F_{22}(\bar{w})J_0(\bar{w})\,dv \quad \text{with } w = w(v), \tag{106b}$$

if the limits exist. By (90)

$$\pi^{-2s}\int_0^T F_{22}(w)J_0(w)\,dv = \int_0^T F_{22}^{+}(w)J_0^{+}(w)\,dv + \int_0^T F_{22}^{-}(w)J_0^{-}(w)\,dv, \quad w = w(v) \tag{107a}$$

and by (90a) & (92)

$$\int_0^T F_{22}^{\pm}(w)J_0^{\pm}(w)\,dv = \sum_n{}^{\circ}\mu(n)\int_0^T F_{22}^{\pm}(w)g(n,w)q_0^{\pm}(n,w;s)\,dv, \tag{107b}$$

where

$$F_{22}(w) := \Gamma(w-s)2^w S_0(w)Z(w) \ \ \& \ \ F_{22}^{\pm} := F_{22}(w)\pi^{-w}e^{\pm\frac{\mathrm{i}\pi w}{2}}, \tag{107c}$$

and s is in $1/2 \le \sigma < 3/4$ as always. Rewriting the last integrals in (107b) as

$$J_{22}^{\pm}(n,T) = \int_0^T r(n,v)e^{\mathrm{i}\phi(n,v)}\,dv, \quad (n \text{ odd \& squarefree}) \tag{108}$$

with

$$\begin{aligned} r(n,v) &:= |F_{22}^{\pm}(w)g(n,w)q_0^{\pm}(n,w;s)| \ \ \& \\ \phi(n,v) &:= \arg F_{22}^{\pm}(w) + \beta(v) + \psi(v) \ \text{ on } N_w \end{aligned} \tag{108a}$$

using (93) and (100) & (104); we have by (85a) & (91) and by (77) & (79)

$$\begin{aligned} r(n,v) &\ll n^{-7/4}e^{\frac{\pi v}{2}}|\Gamma(w-s)S_0(w)Z(w)| \\ &\ll n^{-7/4}(1+|v-t|)^{-h(v)}\log^2(2+v) \\ &\ll n^{-7/4}e^{-c[\log(2+v)]^{1/5}}\log^2(2+v), \\ &\quad \text{uniformly for } n>1 \ \& \ v>1. \end{aligned} \tag{109}$$

Also, from (108a) & (107c) and with (78) & (83a)

$$\phi(n,v) = \gamma(v) - v\log\frac{\pi}{2} \pm \frac{\pi u}{2} + \alpha_0(v) + \alpha(v) + \beta(v) + \psi(v), \tag{110}$$

where

$$\gamma(v) := \arg\Gamma(w-s) \ \text{ on } w = w(v) \text{ is given in (73).} \qquad (110a)$$

Additionally, by (78) for α_0', (83b) for α', (93a) for β' and (104a) for ψ'

$$\begin{aligned}\phi'(n,v) &= \log v - O(1) - o(1) - O(h^{-1}(v)\log\log v) - \log n - o([\log n]^{1/2}) \\ &= \Big[1 - O([\log v]^{-1/6})\Big]\log v - \Big[1 + O([\log n]^{-1/2})\Big]\log n \qquad (111)\end{aligned}$$

for $n \gg 1$ & $v \gg 1$, and the O-estimates are uniform in n & v. This shows that $\phi(n,v)$ is dominated by $\gamma(v)$ and that $\phi'(n,v)$ for $v \gg 1$ is monotonically increasing and so equals zero at most once, say at v^*; where $v^* = n^{1+\nu(n)}$ with $\nu(n) = O([\log n]^{-1/6})$ as $n \to \infty$.

Next we need lower bounds for $|\phi'(n,v)|$ away from v^*. Let $v_1 = v^*(1-\delta)$ and $v_2 = v^*(1+\delta)$ with $\delta = (v^*)^{-1/2} < 1/2$. Taking $N_0 \gg 1$ such that for $n \geq N_0$ & $v_1 \leq v \leq v_2$ the O-terms in (111) are less than $1/4$ and also $\nu(n) < 1/4$, we obtain from (111) in simplified notation, since $\phi'(n,v^*) = 0$

$$\phi'(n,v_1) = (1-O_1)\log v^* + (1-O_1)\log(1-\delta) - (1+O)\log n < 0;$$

i.e.

$$|\phi'(n,v_1)| \geq \frac{1}{2}|\log(1-\delta)| \geq \frac{1}{2}\delta, \quad v_1 = v^*(1-\delta); \qquad (112a)$$

and

$$\begin{aligned}\phi'(n,v_2) &= (1-O_2)\log v^* + (1-O_2)\log(1+\delta) - (1+O)\log n \\ &\geq \frac{1}{2}\log(1+\delta) \geq \frac{1}{3}\delta, \quad v_2 = v^*(1+\delta), \qquad (112b)\end{aligned}$$

where

$$v^*\delta = (v^*)^{1/2} < n^{5/8} \ \text{ and } \ n \geq N_0 \gg 1, \ \ (N_0 \text{ an absolute const.}). \qquad (112c)$$

For $n < N_0$ and $v \geq N_0^2$ say, we obtain from (111), if N_0 is sufficiently large,

$$\phi'(n,v) > \frac{3}{4}\log N_0^2 - O(1) - \frac{5}{4}\log N_0 > \frac{1}{8}\log N_0;$$

i.e.

$$\phi'(n,v) > 1 \ \text{ for } 3 \leq n < N_0 \ \& \ v \geq N_0^2. \qquad (113)$$

The preceding holds for $\phi(n,v)$ of (108a) with odd squarefree n and yields

Lemma 7 *Let* $v_0 = \frac{1}{2}N_0$ *and* $\tilde{v} > N_0^2$ *with a suitable constant* N_0*. Then*

$$\left|\int_{v_0}^{\tilde{v}} e^{\mathrm{i}\phi(n,v)}\,dv\right| < N_0^2 + 22n^{5/8}, \quad \textit{if } \mu(2n) \neq 0. \qquad (114)$$

Proof: First let $n < N_0$. Writing $\phi = \phi(v) = \phi(n, v)$ we have

$$\left|\int_{v_0}^{\tilde{v}} e^{\mathrm{i}\phi}\, dv\right| \le N_0^2 - v_0 + \left|\int_a^{\tilde{v}} e^{\mathrm{i}\phi}\, dv\right| \le N_0^2 - \frac{1}{2}N_0 + \tilde{K} \quad \text{say, with } a = N_0^2$$

and

$$\begin{aligned}\int_a^{\tilde{v}} \cos\phi\, dv &= \int_a^{\tilde{v}} \frac{1}{\phi'(v)} \cos\phi(v)\cdot\phi'(v)\, dv \\ &= \frac{1}{\phi'(a)}\left[\sin\phi(b) - \sin\phi(a)\right], \quad (a < b \le \tilde{v})\end{aligned}$$

by the second mean-value theorem, since here $\phi'(v) > 0$ by (113) and ϕ' increases by (111). The same argument works for the sine-integral and so $\tilde{K} < 4$ (for all $\tilde{v} > N_0^2$), since $\phi'(a) = \phi'(n, N_0^2) > 1$. Now let $n \ge N_0$. Then $v_0 < N_0 - N_0^{1/2} \le n - n^{1/2} < v^* - (v^*)^{1/2} = v_1$ of (112a), and $\phi'(v) = \phi'(n, v)$ remains negative, but increases as v varies from v_0 to v_1 and so does $1/|\phi'(v)|$. Considering for $j = 0, 1, 2$ and with $v_3 = \infty$

$$K_j(x) := \int_{v_j}^{x} e^{\mathrm{i}\phi(v)}\, dv \quad \text{for } v_j \le x \le v_{j+1}, \quad K_j(x) := 0 \text{ elsewhere,}$$

we obtain by the second mean-value theorem

$$\Re e K_0(x) = \int_{v_0}^{x} \frac{-1}{|\phi'(v)|} \cos\phi(v)\phi'(v)\, dv = \frac{1}{|\phi'(x)|}\left[\sin\phi(v_0) - \sin\phi(x)\right]$$

and so by (112a)

$$|\Re e K_0(x)| \le \frac{2}{|\phi'(v_1)|} \le 4\delta^{-1} \quad \text{and then} \quad |K_0(x)| \le 8\delta^{-1} < 8n^{5/8}$$

by (112c). Since $\phi'(v) > 0$ for $v \ge v_2$ and $1/\phi'(v)$ decreases with increasing v, we obtain similarly and with (112b)

$$|K_2(x)| \le \frac{4}{\phi'(v_2)} \le 12\delta^{-1} < 12n^{5/8}. \quad \text{Clearly} \quad |K_1(x)| \le v_2 - v_1 < 2n^{5/8}.$$

Now

$$\left|\int_{v_0}^{\tilde{v}} e^{\mathrm{i}\phi(v)}\, dv\right| \le |K_0(v_1)| + |K_1(v_2)| + |K_2(\tilde{v})| < 22n^{5/8},$$

completing the proof of Lemma 7. □

We also employ

Lemma 8 *Let $r(v)$ and $\rho(v)$ of class $C^1(v_0, \infty)$, $0 \le r(v) < \rho(v)$ and $\rho'(v) < 0$ for $v > v_0 = \frac{1}{2}N_0$; and let $\phi(v)$ in $C^0(v_0, \infty)$ be such that*

$$\left|\int_{v_0}^{x} e^{\mathrm{i}\phi(v)}\, dv\right| < K \quad \textit{for } x > N_0^2. \textit{ Then } \quad \left|\int_{T}^{x} r(v) e^{\mathrm{i}\phi(v)}\, dv\right| < 4K\rho(T) \quad (115)$$

whenever $x > T > N_0^2$. Especially, if also $\rho(v) \to 0$ as $v \to \infty$, then

$$\tilde{J}(T) := \int_T^\infty r(v) e^{\mathrm{i}\phi(v)}\, dv \quad \textit{exists and} \quad |\tilde{J}(T)| \leq 4K\rho(T). \tag{116}$$

Proof: We have for $x > T$

$$\int_T^x r(v) e^{\mathrm{i}\phi(v)}\, dv = r(x)\chi(x) - \int_T^x r'(v)\chi(v)\, dv \quad \text{with} \quad \chi(x) := \int_T^x e^{\mathrm{i}\phi(v)}\, dv. \tag{117}$$

Also

$$\chi(x) = \int_{v_0}^x e^{\mathrm{i}\phi(v)}\, dv - \int_{v_0}^T e^{\mathrm{i}\phi(v)}\, dv \quad \text{yields} \quad |\chi(x)| < 2K, \quad \text{if } T > N_0^2.$$

We introduce for $v > N_0^2$

$$r_0(v) := \left\{ \begin{matrix} r(v), \text{ if } C^* \\ 0, \text{ if C} \end{matrix} \right\}, \; r_1(v) := \left\{ \begin{matrix} 0, \text{ if } C^* \\ \rho(v) - r(v), \text{ if C} \end{matrix} \right\}, \; r_2(v) := \left\{ \begin{matrix} 0, \text{ if } C^* \\ \rho(v), \text{ if C} \end{matrix} \right\},$$

where C^* resp. C stands for $r'(v) < 0$ resp, $r'(v) \geq 0$. Then

$$r(v) = r_0(v) - r_1(v) + r_2(v) \quad \text{and} \quad r_j(v) \geq 0, \; r_j'(v) \leq 0 \quad \text{for } j = 0, 1, 2.$$

Replacing $r(v)$ by $r_j(v)$ in (117) yields

$$\begin{aligned} \left| \int_T^x r_j(v) e^{\mathrm{i}\phi(v)}\, dv \right| &\leq r_j(x)|\chi(x)| + \int_T^x |r_j'(v)| \cdot |\chi(v)|\, dv \\ &\leq 2K \left(r_j(x) - \int_T^x r_j'(v)\, dv \right) \;=\; 2K r_j(T), \end{aligned}$$

and so

$$\left| \int_T^x r(v) e^{\mathrm{i}\phi(v)}\, dv \right| \leq 2K \sum_{j=0}^{2} r_j(T) \leq 4K\rho(T) \quad \text{for } x > T.$$

□

8. Considering again the integral in (108) we note that by (109)

$$r(n, v) < \rho(v) = \rho(n, v) := An^{-7/4}[\log(2 + v)]^{-4}, \quad A = \text{const.} \tag{118}$$

say. Using Lemma 8 with $r(v) = r(n, v)$ and $K = K(n) := N_0^2 + 22n^{5/8}$ by Lemma 7, we infer for $T > N_0^2$ that the complementary integral

$$\tilde{J}(n, T) := \int_T^\infty r(n, v) e^{\mathrm{i}\phi(n,v)}\, dv \;\; \text{exists and} \;\; |\tilde{J}(n, T)| \leq 4K(n)\rho(n, T). \tag{119}$$

From (107b) with the notation of (108) we now obtain

$$\int_0^T F_{22}^{\pm}(w) J_0^{\pm}(w)\, dv = \sum_n{}^{\circ} \mu(n) \left[\int_0^\infty r(n, v) e^{\mathrm{i}\phi(n,v)}\, dv - \tilde{J}(n, T) \right], \tag{120a}$$

and

$$\sum_n^\circ |\mu(n)\tilde{J}(n,T)| \quad < \quad 4A[\log(2+T)]^{-4}\sum_{n>0}(N_0^2+22n^{5/8})n^{-7/4}$$
$$< \quad \tilde{A}[\log(2+T)]^{-4} \to 0 \ \text{ as } T\to\infty. \qquad (120b)$$

Hence the limits

$$\lim_{T\to\infty}\int_0^T F_{22}^{\pm}(w)J_0^{\pm}(w)\,dv = \sum_n^\circ \mu(n)\int_0^\infty F_{22}^{\pm}(w)g(n,w)q_0^{\pm}(n,w;s)\,dv \quad (121)$$

exist, and then by (107a) the first limit in (106b) exists and so does the second limit there, by an analogous calculation; i.e. $J_{22}^* = J_{22}^*(s)$ actually exists.

Regarding the dependence on the parameter s, which we often omitted in our notation, we remark that s appears only in the arguments of analytic functions and that the estimates used in the preceding convergence proof are uniform for s on $1/2 \le \sigma \le 3/4$ & $|t| \le T$ and arbitrary $T \gg 1$. Hence $J_{22}^*(s)$ is continuous there and holomorphic on $1/2 < \sigma < 3/4$.

It is visible that the first and the third integral in (84b), denoted by J_{21}^* and J_{23}^* say, can be treated the same way as J_{22}^* following (106a,b). This yields the existence of

$$J_2^* = J_{21}^* - 2J_{22}^* + J_{23}^* - J_3^* \ \text{ with } \ J_3^* = J_{31}^* - J_{32}^* \qquad (122)$$

by (84c), since for instance the last integral there becomes

$$J_{32}^* \quad = \quad \int_{N_w}\Gamma(w-s)2^w Z(w)J_0(w)\left[\frac{\Gamma'}{\Gamma}(w-s)+\log 2\right]dw$$
$$+\int_{N_w}\Gamma(w-s)2^w Z(w)J_0'(w)\,dw, \qquad (123)$$

and here the first integral exists by (76) and comparison with J_{23}^* in (84b), while the second integral exists using (91a), (105) & (105a) and arguing as for J_{22}^* above. The next to last integral J_{31}^* in (84c) exists similarly. Finally, convergence of the integrals J_0^* and J_1^* of (84a) follows with (90) & (90a) for $k=2$ & 1 in the same way as for J_{22}^*, using (87a), (96) and (99) additionally, to establish the crucial estimates corresponding to (109) & (111).

This completes the proof of convergence of the integrals $J_k^* = J_k^*(s)$, $(k = 0,1,2)$ in (71a) and implies their continuity and holomorphy, as for J_{22}^* above. By the remarks surrounding (71a) we have thereby proven our assumption A* and the existence of the iterated integral $H(s)$ in (53); which implies with (65) that $\lim_{\epsilon\to 0} J(s,\epsilon) = H(s)$ for s in $1/2 \le \sigma \le 3/4$, as we have seen. Since now we have $2\pi \mathrm{i} H(s) = J_0^*(s) + J_1^*(s) + J_2^*(s)$, $H(s)$ also is holomorphic on $1/2 < \sigma < 3/4$ and continuous on $1/2 \le \sigma \le 3/4$; i.e. our proof of the Main Lemma is complete; and in conclusion, Theorems 1 & 2 are validated.

13 Acknowledgement

I thank my longtime friends and colleagues John G. Ratcliffe and Steven T. Tschantz for their trust and for diligently transforming my handwritten work into a TEX- written print-ready manuscript.

14 Postscript

Responsibility for the contents of this paper rests solely with the author, and any correspondence about it should be sent to him: Prof. Dr. R. Arenstorf, 4633 Benton Smith Rd., Nashville, TN37215; USA.

References

[1] K. Chandrasekharan, *Arithmetical Functions*, Springer-Verlag, New York, 1970.

[2] J. Korevaar, A Century of Complex Tauberian Theory, *Bull. Amer. Math. Soc.* **39**, No. 4 (2002), 475-531.

[3] K. Prachar, *Primzahlverteilung*, Springer-Verlag, Berlin, 1957.

[4] E.C. Titchmarsh, *The Theory of the Riemann Zeta Function*, Oxford Univ. Press, Oxford, 1951.

[5] P. Turan, Some Function-Theoretic Sieve Methods in the Theory of Numbers, *Soviet Math. Doklady*, **7**, No. 6, (1966), 1661-1664.

[6] P. Turan, On the Twin-Prime Problem II, *Acta Arithmetica* **13**, (1967), 61-89.

[7] D.V. Widder, *The Laplace Transform*, Princeton Univ. Press, Princeton, 1941.

Department of Mathematics, Vanderbilt University, Nashville, TN 37240; USA